Comment on "Synchronization in a ring of four mutually coupled van der Pol oscillators: Theory and experiment"

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We have verified some results of Nana and Woafo [Phys. Rev. E **74**, 046213 (2006)] in the area of the complete synchronization. We have found that the motion of the van der Pol network is quasiperiodic, not chaotic as the authors have written.

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The complete synchronization (CS) in the ring of van der Pol (VDP) oscillators is a well-known fact, it has been investigated in many papers but mostly numerically. That is why the experimental and numerical studies shown in [1] are very interesting and have focused our attention. Here we comment on some results of this paper.

In [1] the authors considered the ring of four mutually coupled VDP oscillators. They introduced the following dimensionless equation of motion:

$$\ddot{x}_1 - d_1(1 - x_1^2)\dot{x}_1 + x_1 = k_1(x_4 + 2x_1 - x_2) + F_1 \cos(\Omega t),$$

$$\ddot{x}_2 - d_2(1 - x_2^2)\dot{x}_2 + x_2 = k_2(x_3 + 2x_2 - x_1) + F_2 \cos(\Omega t),$$

$$\ddot{x}_3 - d_3(1 - x_3^2)\dot{x}_3 + x_3 = k_3(x_2 + 2x_3 - x_4) + F_3 \cos(\Omega t),$$

$$\ddot{x}_4 - d_4(1 - x_4^2)\dot{x}_4 + x_4 = k_4(x_1 + 2x_4 - x_3) + F_4 \cos(\Omega t). \tag{1}$$

In Sec. III C ([1]) the authors mention the frequency of the driving signal as $\Omega = 0.35$ Hz, but we think that this value corresponds to the dimensionless driving frequency ($\frac{\Omega}{\omega}$) and there should be no Hz.

The authors quite often repeat the word *chaotic* in the article. In Fig. 8 ([1]) there is a phase plane of the VDP before coupling, but for us it looks similar to a quasiperiodic and motion not the chaotic one. To prove our assumption, in Fig. 1 we present a phase plane [Fig. 1(a)] and a Poincare section [Fig. 1(b)] for the same values of parameters as in the commented article (we have not found the dimensionless frequency of the excitation signal, so we take the value 0.35, as in the experimental investigations). As can be easily seen, the Poincaré section is a closed curve (a section of the tori), also the values of Lyapunov exponents indicate a quasiperiodic motion ($\lambda_1 = 0.000, \lambda_2 = -0.204$). We have been also searching for a chaotic behavior in a wider range of control parameters [changing frequency (Ω) and amplitude (F) of the driving signal—see Eq. (1)] without success. The same situation takes place while checking the results from their Fig. 2 [1].

We discuss, in more details, their Fig. 2 [1]. In our numerical investigations we have not found an area of desynchronization for $F > 2.2$. We have based our calculations on two independent algorithms. The firstone is the master stability function (MSF) introduced by Pecora [2]. This concept is based on transversal Lyapunov exponents (TLEs) and it is very useful in investigations of identical system networks. The second algorithm is a typical calculation of the synchronization error. In Fig. 2 we show the synchronization error z [Fig. 2(a)], MSF [Fig. 2(b)], and its enlargements [Figs. 2(c) and 2(d)] for four mutually coupled VDP oscillators, with the same values of parameters as in thier Fig. 2 [1]. The white color in all figures corresponds to the CS range, the black color consequently corresponds to the lack of CS. As can be seen, the two plots are identical, so it proves our assumption. On the other hand, if the coupling strength is sufficient and the topology of connections between the systems is adequate, then nonlinear systems can synchronize in any state—chaotic, quasiperiodic, or periodic, and there is no necessity to favor one of them. The synchronization between self-

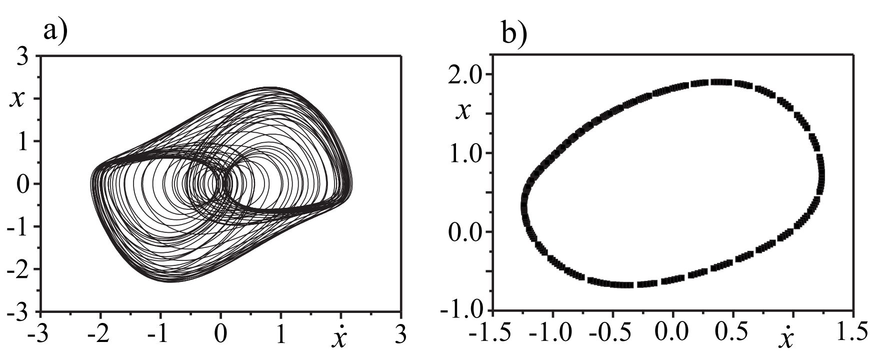

FIG. 1. Quasiperiodic motion of the VDP oscillator: (a) Phase plane and (b) Poincaré section.

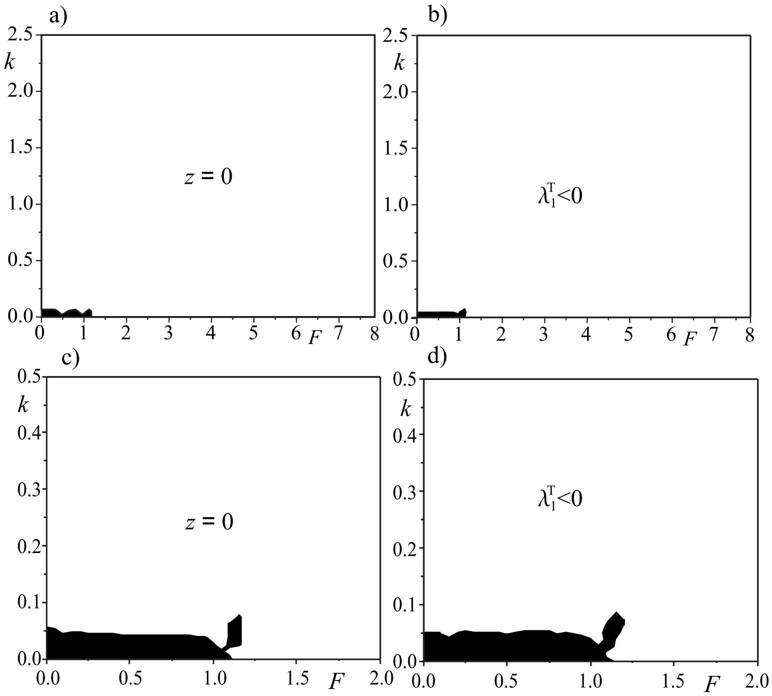

FIG. 2. CS in the network of four coupled VDP oscillators; $k_1=k_2=k$. (a) Synchronization error z, (b) MSF, (c) enlargements of the synchronization error z, and (d) MSF. White color: $z=0$ or $\lambda_1^T<0$; black color: $z>0$ or $\lambda_1^T>0$.

sustained oscillators in the periodic or quasiperiodic range is often associated with the mode locking with excitation. So if the driving signal is the same for all coupled systems, the CS is a quite common state. That is why the disappearance of the CS, in such a wide range of parameters, is quite strange.

We have built the circuit using the scheme shown in the discussed article. We have reached a good agreement of the experimental and numerical investigations [3].

[1] B. Nana and P. Woafo, Phys. Rev. E **74**, 046213 (2006).
[2] L. Pecora and T. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[3] P. Perlikowski, A. Stefanski, and T. Kapitaniak (unpublished).