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Periodization of Duffing oscillators suspended on elastic structure: Mechanical explanation

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Abstract

We consider the dynamics of chaotic oscillators suspended on the elastic structure. We show that for the given conditions of the structure, initially uncorrelated chaotic oscillators can synchronize both in chaotic and periodic regimes. The phenomena of the periodization, i.e., the behavior of nonlinear oscillators become periodic as a result of interaction with elastic structure, have been observed. We formulate the criterion for periodization of double well-potential Duffing oscillator evolution in terms of the forces and displacements in the spring elements. We argue that the observed phenomena are generic in the parameter space and independent of the number of oscillators and their location on the elastic structure.

1. Introduction

The phenomenon of synchronization in dynamical and, in particular, mechanical systems has been known for a long time. In the last decade of the XX century the idea of synchronization has been adopted for chaotic systems [1–13]. It has been demonstrated that two or more chaotic systems can synchronize by linking them with mutual coupling or with a common signal or signals. In the case of linking a set of identical chaotic systems (the same set of ODEs and values of the system parameters) complete synchronization can be obtained. The complete synchronization takes place when all trajectories converge to the same value and remain in step with each other during further evolution (i.e., $\lim_{t\to\infty}|x(t) - y(t)| = 0$ for two arbitrarily chosen trajectories $x(t)$ and $y(t)$). In such a situation all subsystems of the augmented system evolve on the same manifold on which one of these subsystems evolves (the phase space is reduced to the synchronization manifold). Linking homochaotic systems (i.e., systems given by the same set of ODEs but with different values of the system parameters) can lead to imperfect synchronization (i.e., $\lim_{t\to\infty}|x(t) - y(t)| \leqslant \varepsilon$, where ε is a vector of small parameters) [14] or to the phase synchronization [15]. In such linked systems it can be also observed a significant change of the chaotic behavior of one or more systems.

In the current studies we consider the synchronization of nonlinear chaotic oscillators located on (coupled through) elastic structure. We present a numerical study of a realistic model of two double well-potential Duffing oscillators

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suspended on the elastic beam. In both cases nonlinear oscillators are externally excited by periodic force with a frequency ω.

We show that for the given conditions of the elastic structure (linear oscillator or elastic beam), initially uncorrelated chaotic oscillators can synchronize both in chaotic and periodic regimes. One can observe the phenomena of the periodization of oscillators, i.e., the behavior of nonlinear oscillators becomes periodic as a result of interaction with elastic structure.

We argue that the observed phenomena are generic in the parameter space and independent of the number of oscillators and their location on the elastic structure. Additionally, we give the physical explanation of the observed phenomena in terms of the forces and displacements in the spring elements.

The paper is organized as follows. In Section 2 we recall some fundamental properties of chaotic behavior in double well-potential Duffing oscillator. Section 3 considers dynamics of Duffing oscillator connected with a linear oscillator and forms the criterion for periodization of the oscillator's evolution. The behavior of two Duffing oscillators suspended on the elastic beam is discussed in Section 4. Finally our results are summarized in Section 5.

2. Chaos in the single oscillator

We consider double well-potential Duffing oscillator shown in Fig. 1. Its evolution is described by:

$$m\ddot{y} + d_y\dot{y} - k_y y + k_d y^3 = F\sin\omega t \tag{1}$$

where m, d_y, k_y, k_d, F and ω are constant. Oscillator (1) has three equilibria: unstable for $y_{00} = 0$ and two stable for $y_{01} = 1$ and $y_{02} = -1$. In our numerical analysis we assumed $m = 1$, $d_y = 0.168$, $k_y = 0.5$, $k_d = 0.5$, $F = 0.21$ and $\omega = 1$, i.e., oscillator (1) shows chaotic behavior [16,17].

In Fig. 2a we have plotted time series (y versus t), where time t is described by a number N of periods of excitation $2\pi/\omega$ and in Fig. 2b the adequate phase portrait is shown. One can observe that the necessary condition for the occurrence of chaotic behavior are the trajectory jumps between neighborhoods of stable equilibria y_{01}, and y_{02}. Such jumps occur when the difference between $|y - y_{01,02}|$ is larger than 1. The decrease of the value of excitation amplitude F leads to $|y - y_{01,02}| < 1$ and periodic oscillations in the neighborhood of y_{01} or y_{02}.

3. Periodization of double well-potential duffing oscillator: linear analogy

Now we can form a question if the periodization of the behavior of Duffing oscillator (1) can be achieved by connecting it (in series) to the other linear oscillator. After such a connection the spring length $|y - y_{01,02}|$ is reduced to $|y - z - y_{01,02}|$, where z is the displacement of a linear oscillator. Reducing $|y - z - y_{01,02}|$ to value less than 1 means that Duffing oscillator does not leave the neighborhood of one's stable equilibria hence cause of chaotic behavior can be eliminated. The optimal system in respect of the minimization of $|y - z - y_{01,02}|$ is the linear oscillator in the resonance range.

This phenomenon can be explained on the example of a linear undamped system shown in Fig. 3a. The evolution of the system is described by:

$$\begin{aligned} m\ddot{y} + k_y(y - z) &= F\sin\omega t \\ u\ddot{z} + k_y(z - y) + k_z z &= 0 \end{aligned} \tag{2}$$

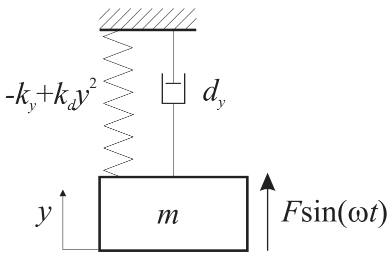

Fig. 1. Double well-potential Duffing oscillator.

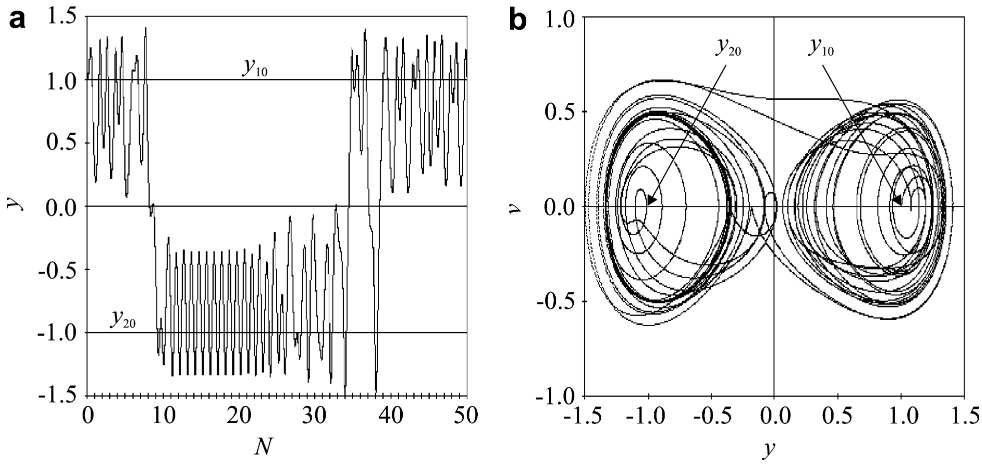

Fig. 2. Chaotic evolution of Duffing oscillator (1); $m = 1$, $d_y = 0.168$, $k_y = 0.5$, $k_d = 05$, $F = 0.21$ and $\omega = 1$: (a) time series and (b) phase space.

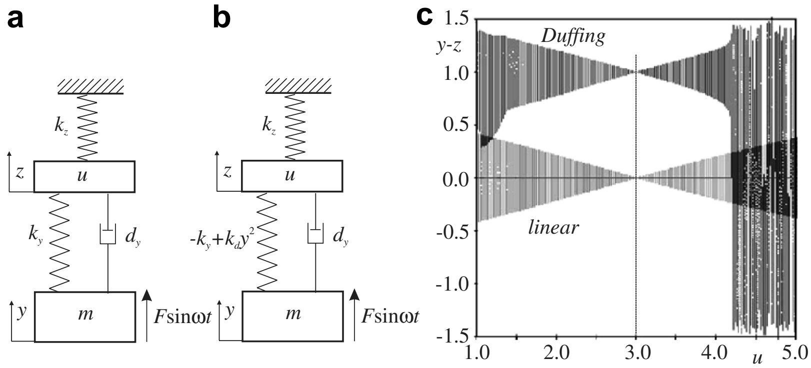

Fig. 3. (a,b) Two coupled (in series) oscillators and (c) bifurcation diagram $y - z$ versus u, linear oscillator (shown in grey): $m = u = 1$, $d_y = 0$, $k_y = 1$, $k_d = 0$, $k_z = \omega^2 u$, $F = 0.21$, $\omega = 1$, Duffing oscillator (shown in black): $m = u = 1$, $d_y = 0.168$, $k_y = 0.5$, $k_d = 0.5$, $F = 0.21$, $\omega = 1$, $k_z = \omega^2 u$.

and the system (2) has one equilibrium $y = z = 0$. For $k_z = \omega^2 u$ amplitudes of two masses (u and m) are equal. Mass u oscillates with amplitude $z_{\max} = \frac{F}{-\omega^2 m}$ and driving force F moves in mass m in such a way that two oscillators behave in the same manner, i.e., $y = z$. There is no force in spring k_y so this spring has no influence on the evolution of the system (2).

The bifurcation diagram of spring displacement $y - z$ versus mass u has been shown in Fig. 3c. In numerical calculation we have taken: $m = u = 1$, $k_y = -1$, $d_y = 0$, $k_z = \omega^2 u$, $k_d = 0$, $F = 0.21$, $\omega = 1$, in the case of linear oscillator and $m = u = 1$, $d_y = 0.168$, $k_y = 0.5$, $k_d = 0.5$, $F = 0.21$, $\omega = 1$, $k_z = \omega^2 u$ in the case of Duffing oscillator.

One can see that changing u it is possible to reduce spring displacement even to zero (for $u = 3$). This phenomena has been observed not only for the linear oscillator (in this case it be explained analytically) but for Duffing oscillator as well. Results for linear and Duffing oscillators are shown respectively in gray and black in Fig. 3c.

4. Duffing oscillators suspended on the elastic beam

Consider the system, shown in Fig. 4. Two identical Duffing oscillators are suspended on two concentrated masses u_1 and u_2 located on the massless elastic beam. The evolution of the system is described by

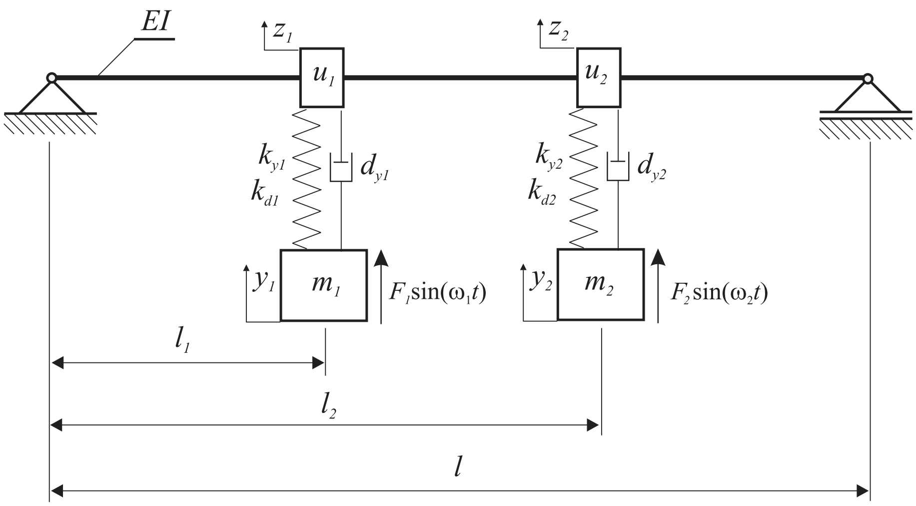

Fig. 4. Two Duffing oscillators suspended on the elastic beam.

$$\begin{aligned}
&m_1\ddot{y}_1 + d_{y1}(\dot{y}_1 - \dot{z}_1) + k_{y1}(y_1 - z_1) + k_{d1}(y_1 - z_1)^3 = F_1 \sin \omega_1 t \\
&m_2\ddot{y}_2 + d_{y2}(\dot{y}_2 - \dot{z}_2) + k_{y2}(y_2 - z_2) + k_{d2}(y_2 - z_2)^3 = F_2 \sin \omega_2 t \\
&u_1\ddot{z}_1 + d_{11}\dot{z}_1 + k_{11}z_1 + k_{12}z_2 - d_{y1}(\dot{y}_1 - \dot{z}_1) - k_{y1}(y_1 - z_1) - k_{d1}(y_1 - z_1)^3 = 0 \\
&u_2\ddot{z}_2 + d_{22}\dot{z}_2 + k_{21}z_1 + k_{22}z_2 - d_{y2}(\dot{y}_2 - \dot{z}_2) - k_{y2}(y_2 - z_2) - k_{d2}(y_2 - z_2)^3 = 0,
\end{aligned} \tag{3}$$

where k_{ij} is the coefficient of stiffness matrix of the beam and d_{ij} is the coefficient of damping matrix of the beam. In numerical investigations we have taken the following values of system parameters:

(i) Duffing oscillators:
$m_{1,2} = 1.0, d_{y1,2} = 0.168, k_{y1,2} = -0.5, k_{d1,2} = 0.5, F_{1,2} = 0.21, \omega_{1,2} = 1.0,$

(ii) Elastic beam:
$l = 1.0$, $l_1 = 0.333$, $l_2 = 0.666$, $EI = 1/48$ (with such a stiffness of the natural frequency of oscillations of the beam with one mass $u = 1.0$ located at the half is equal to 1 and equal to the natural frequency of suspended oscillator $k_d = 0$, $k_y = -1$,); E is the modulus of elasticity of the beam material and I is the moment of inertia of the beam cross section about its central line, stiffness coefficient of the beam k_{ij} were computed from the equation of beam deflection $EIy'' = M_g$. Damping coefficient d_{11} and d_{22} are ratios of matrix u, where $u = \mathrm{diag}[u_1,u_2]$. The coupling introduced in the system (3) is the example of nondiagonal coupling which is typical for mechanical systems [8,9].

As we have mentioned, the investigated oscillator can jump from the neighborhood of one stable equilibrium to the neighborhood of another one. This jumps are responsible for chaotic behavior. As described in Section 3 the periodization of oscillations in system (3) can be achieved by decreasing the length of the spring $(y - z)$ without changing the amplitude y. Consider a linear beam with two nonlinear oscillators. If the stiffness of the beam EI is too big then the displacement z is too small to change the character of the oscillator's behavior so we still have chaotic behavior. A suitable reduction of the beam stiffness changes the character of the oscillator behavior to the periodic one as can be seen in Fig. 5 where we have shown the plots of y_1 and z_1 versus time t for beam stiffness $EI = 1/48$ and different masses of u_1 and u_2. It is worth to notice that due to the symmetry of the system (3) two Duffing oscillators synchronized, i.e., $y_1 = y_2$.

Small asymmetry of the system has no influence on it. Oscillations of Duffing systems take place in the neighborhood of state of equilibrium $y_{01} = 1$, to facilitate assessment we have shown value y_1 reduced by 1.

As we have shown in Fig. 5a masses $u_{1,2}$ are small enough, there is no phase shift between displacement of the oscillator and mass settled on the beam, the system is in phase synchronization range. The beam is following the evolution of Duffing oscillators reducing the length of the springs and changing oscillators behavior to the periodic motion. Increase of masses $u_{1,2}$ to the value $u_{1,2} > 0.65$ results in the fact, that the natural frequency of the beam is equal to 1, i.e., to frequency of driving excitation. In Fig. 5b we have shown courses of displacements $y_1 - 1$ and z_1 are nearly identical

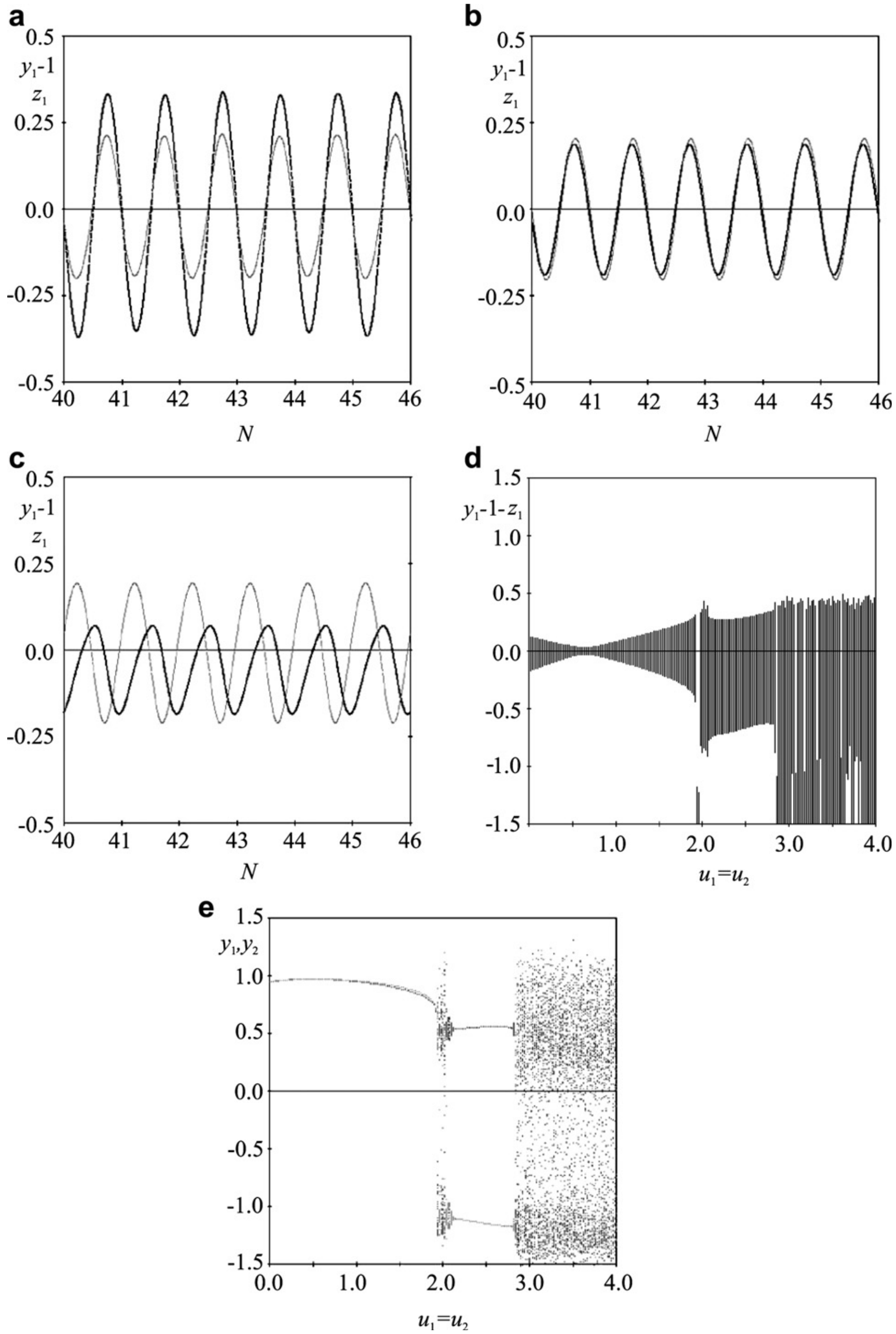

Fig. 5. Evolution of the system (3); $m = 1.0$, $d_y = 0.168$, $k_y = -0.5$, $k_d = 0.5$, $F = 0.21$, $\omega = 1.0$, $l = 1.0$, $l_1 = 0.333$, $l_2 = 0.666$, $EI = 1/48$: (a) $u_1 = u_2 = 0.005$, (b) $u_1 = u_2 = 0.8$, (c) $u_1 = u_2 = 1.8$, (d) bifurcation diagram $y_1 - 1 - z_1$ versus $u_1 = u_2$ and (e) bifurcation diagram y_1, y_2 versus $u_1 = u_2$.

(in Section 3 we described these phenomena in the example of two linear oscillators). With further increase of masses $u_{1,2}$, the phase shift between Duffing oscillators and beam appear as it is shown in Fig. 5c for $u_{1,2} = 1.8$. In this case elastic beam has already passes the resonance and one can see that displacements amplitudes of the springs are larger.

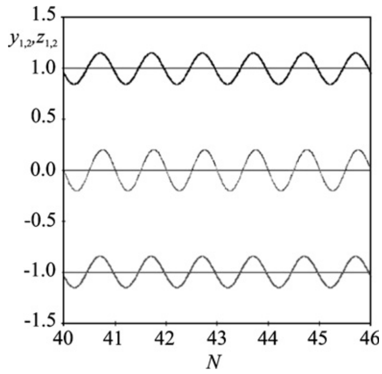

Fig. 6. Evolution of the system (3) when Duffing oscillators operate on different periodic attractors; $m = 1.0$, $d_y = 0.168$, $k_y = -0.5$, $k_d = 0.5$, $F = 0.21$, $\omega = 1.0$, $l = 1.0$, $l_1 = 0.333$, $l_2 = 0.666$, $EI = 1/48$.

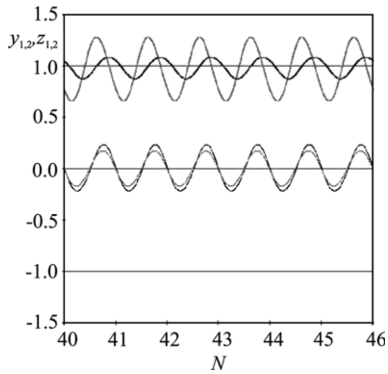

Fig. 7. Evolution of the system (3) in the case of asymmetrical suspension of Duffing oscillators; $m = 1.0$, $d_y = 0.168$, $k_y = -0.5$, $k_d = 0.5$, $F = 0.21$, $\omega = 1.0$, $l = 1.0$, $l_1 = 0.5$, $l_2 = 0.75$, $EI = 1/48$.

In Fig. 5d the bifurcation diagram: displacement $y_1 - 1 - z_1$ of the spring versus masses $u_{1,2}$ is shown. One can see that the minimum of the oscillations amplitude occurs for $u_{1,2} = 0.65$. Fig. 5e shows bifurcation diagram of y_1 and y_2 versus $u_1 = u_2$. One can observe that for masses $u_{1,2} < 1.8$ evolution of the system (3) is periodic (as indicated by single dots in Fig. 5d). For larger $u_{1,2}$ ($u_{1,2} > 1.9$) system (3) shows chaotic behavior – two oscillators jumps from one state of equilibrium to another one. Moreover, their motion is not identical because of asymmetry of the system. On the boundary of periodic and chaotic ranges we have coexistence of various attractors (periodic with different period and chaotic ones).

One of the properties of nonlinear oscillators is the coexistence of different attractors in the phase space. For our next example we have chosen initial conditions of two Duffing oscillators in neighborhoods of different equilibria (for one oscillator in the neighborhood of $y_{01} = 1$ and for another one in the neighborhood of $y_{02} = -1$). As we have shown in Fig. 6 periodization of the behavior can be observed. In this case oscillator evolves on different periodic attractors.

Second problem we have taken into consideration is the asymmetry of the points of fixing oscillators to the beam. For numerical example we have chosen $l_1 = 0.5$ and $l_2 = 0.75$ and as it is shown in Fig. 7 periodization of oscillators behavior can also be observed in this case.

5. Conclusions

We investigated the possibility of the synchronization of nonlinear chaotic oscillators located on (coupled through) elastic structure. In the numerical study of a realistic model of two double well-potential Duffing oscillators suspended on the elastic beam we showed that for the given conditions of the elastic structure oscillations initially uncorrelated chaotic oscillators can synchronize in periodic (complete synchronization) regime. We identified the phenomena of the periodization of oscillators in which the behavior of nonlinear oscillators becomes periodic as a result of the interaction with the elastic structure and gave physical explanation of the observed phenomena in terms of the forces and displacements in the spring elements.

We gave numerical evidence that the observed phenomenon is generic in the parameter space and independent of the number of oscillators and their location on the elastic structure.

The problem of the mismatch of Duffing oscillators parameters will be considered elsewhere [18].

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