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Multistability and rare attractors in van der Pol - Duffing oscillator

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We discuss the mechanism leading to the multistability in the externally excited van der Pol - Duffing oscillator. It has been shown that the mechanism (the sequence of bifurcations) leading to the phase multistability in coupled oscillators is the same as the mechanism leading to the bistability in the single oscillator.

Keywords: Rare attractors, multistability, van der Pol - Duffing oscillator

Nonlinear systems exhibit a rich variety of different long-term behaviors such as: fixed points, limit cycles, quasiperiodic and chaotic behavior. In a complex system several attractors may coexist for a given set of system parameters. This coexistence is termed *multistability* and has been found in almost all research areas of natural science, such as: mechanics, electronics, biology, environmental science and neuroscience [Atteneave, 1971; Knorre et al., 1975; Arecchi et al. 1982; Feudel, 2008; Shrimali et al, 2008; Lysyansky et al., 2008; Perlikowski et al., 2010]. Multistable systems are characterized by a high degree of complexity in dynamical behavior due to the interaction among the co-existing attractors.

Generally, in the multistable systems one observes the interaction among the attractors. First, the dynamics of a multistable system is extremely sensitive to initial conditions. Due to the coexistence of different attractors and complex fractal basin boundary structures very small perturbations of the initial state may influence the final attractor. Second, the qualitative behavior of the system often changes under the variation of the system parameters as some attractors exist only in the small intervals of the system parameters. A slight change in a control parameter may cause a rapid change in the number and type of coexisting attractors. Third, multistable systems are extremely sensitive to noise. Noise may cause a popping process between various attractors.

In real, particularly engineering systems the transitions to multistability may cause a disaster, as usually only one of co-existing attractors guarantees the desired behavior of the systems. In connection with this, the challenge to be solved is the prediction of disasters and the abatement of their consequences [Blekhman and Kuznetsova, 2008]. In this paper we discuss this problem in the terms of the concept of *rare attractors* introduced by Zakrzhevsky et al. [2007]. We try to propose the definition of rare attractors and describe their creation in the forced van der Pol - Duffing oscillator.

Consider the dynamical system $\dot{\mathbf{x}} = f(\mathbf{x}, \omega)$, where $\mathbf{x} \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$ is the system parameter. Let $\mathcal{B} \subset \mathbb{R}^n$ be a set of all possible initial conditions and $\mathcal{C} \subset \mathbb{R}^n$ be a set of accessible system parameters. Assume that attractor $\mathcal{A}$ exists for $\omega \in \mathcal{C}_{\mathcal{A}} \subset \mathcal{C}$ and has a basin of attraction (BA) $\beta(\mathcal{A})$. Assuming the initial conditions and system parameters are chosen independently, the probability that the system is on the attractor $\mathcal{A}$ is equal to

$$p(\mathcal{A}) = \frac{\mu(\mathcal{C}_{\mathcal{A}})}{\mu(\mathcal{C})} \frac{\mu(\beta(\mathcal{A}))}{\mu(\mathcal{B})}, \tag{1}$$

where μ is a set measure. If $p(\mathcal{A})$ is small (i.e., $p(\mathcal{A}) << 1$) the attractor $\mathcal{A}$ is called the rare one.

The problem of the estimation of a set of possible initial conditions is commonly met in real life systems. For example, the problem of the uncertainty of the initial condition in electrical systems was taken into account in [Tolsa and Salichs, 1993]. They show that due to the property of circuit related to charge conservation in capacitive cutsets and flux conservation in inductive loops the initial conditions applied to the oscillator are changed for initial time $t = 0$, so actually the circuit starts with other initial conditions than the applied one. This property causes that one has always to take some subset of possible initial conditions which we define as $\mathcal{B}$. One should notice here that the probability that the system is on the attractor $\mathcal{A}$ given by (1) depends on the definitions of the sets $\mathcal{B}$ and $\mathcal{C}$.

As an example of the system which possesses multistability and rare attractors we consider an externally exited van der Pol-Duffing oscillator

$$\ddot{x}(t) - \alpha(1 - x(t)^2)\dot{x}(t) + x(t)^3 = F \sin \omega t, \tag{2}$$

where α, ω and F are positive constants. In what follows we assumed $\alpha = 0.2$, $F = 1.0$ and considered ω as a control parameter.

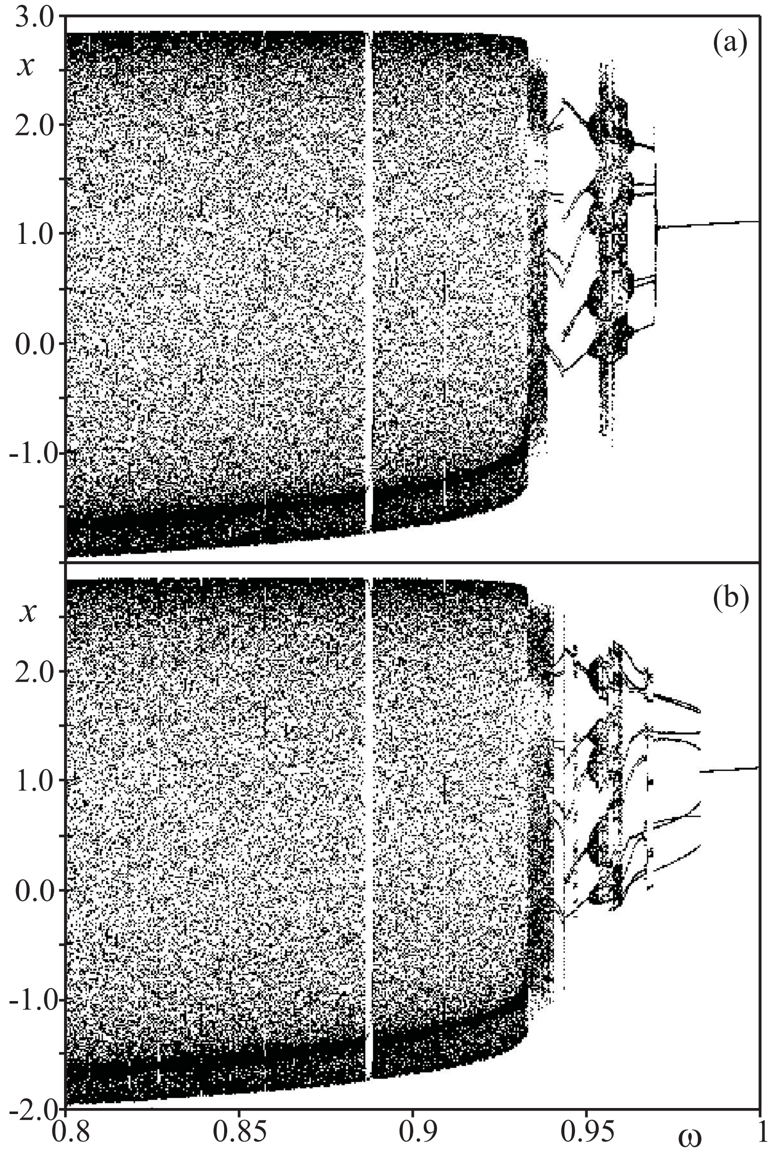

Fig. 1: Bifurcation diagrams for (a) decreasing of external frequency of excitation and (b) decreasing in range $\omega \in [0.8, 1.0]$. In both diagrams numerous different attractors can be identified

Oscillator (2) is known to have the complex dynamical behavior in the neighborhood of the principal resonance, i.e., for $\omega \approx 1$ [Szemplinska-Stupnicka and Rudowski, 1994, 1997; Venkatesan and Lakshmanan, 1997; Anishchenko et al., 2007]. Here, we show that some of the attractors which exist in this neighborhood can be considered as rare attractors.

In Fig. 1(a,b) we present the bifurcation diagrams of system (2), along the frequency ω, on the horizontal axes we plot the values of displacement x which is taken once per period of the excitation. We use four-order Runge-Kutta algorithm for numerical integrations. The results shown in Fig. 1(a) have been calculated for the increasing values of ω while in the second one (Fig. 1(b)) for the decreasing values of

ω. Comparing both diagrams one can notice the large range of multistability for $\omega \in (0.93, 0.98)$. We have carefully checked all the co-existing periodic solutions (PS) and we have detected $T-$, $7T-$, $9T-$, $11T-$, $14T-$, $21T-$ and $35T-$PS, where $T = 2\pi/\omega$. All PS mentioned by us correspond to the mode locked solution which are parts of Arnold's tongs [Simonet et al., 1994]. One can observe the bifurcations of these PS leading through Neimark-Sacker and period doubling bifurcations to the chaotic attractors. All solutions presented in Fig. 1 are stable and due to multistability and fractal BA one can observe the jumps from one attractor to another. To understand the dynamics of the considered system it is necessary to find all or most of the co-existing PS. To do this we use the path-following package AUTO-2000 [Doedel et al., 2002] and the command line interface Rauto [Schilder, 2007]. The higher dimensional attractors are bifurcating from this PS.

The profiles of the PS which serve as initial data for the continuation in AUTO were obtained by numerical integration. We present the results in Fig. 2(a-g), where stable and unstable branches of PS are shown in green and red respectively. The symbols: dot, square, circle and cross correspond to SD (saddle-node) bifurcation, PD (period-doubling) bifurcation, NS (Neimark-Sacker) bifurcation and BF (branching point - by this name AUTO indicate the unknown bifurcation) respectively. The T-PS is stable till $\omega = 0.971$. For this value the NS bifurcation occurs and the stable $2D$ torus is born (Fig. 2(a)). Then, we systematically followed other locked PS. The $7T-$PS exists in $\omega \in (0.909, 0.983)$ (Fig. 2b), for $\omega = 0.951$ and $\omega = 0.964$ we observe supercritical NS bifurcations (see Fig. 1(a,b)), the rest of the bifurcations are SN where the stable and unstable PS meet. In Fig. 2(c) one can see the branches of $9T-$PS, this solution exists in the range $\omega \in (0.925, 0.958)$. After destabilization through the cascade of PD bifurcations (see Fig. 2(d)) one can observe the stable branches of $18T-$PS, $36T-$PS and so on. The next plot is devoted to $11T-$PS which exists for $\omega \in (0.924, 0.986)$. For $\omega = 0.971$ one can observe a PD route to chaos. In Fig.2(f) we show the $14T-$PS which exist between $\omega = 0.91$ and $\omega = 0.945$. As it is easy to see there are twelve NS bifurcations which lead to the appearance of the toruses. In the last plot we show $21T-$PS; there are only two sharp intervals of stable motions, which through saddle-node bifurcation become unstable. We present most of PS and their bifurcations, the higher locking values (like i.e.$35T-$PS) are over the calculation limit of AUTO (the crucial point is preparing initial data for continuation). By looking at these branches one can see that in the considered range of ω there are many co-existing stable attractors and as we show later

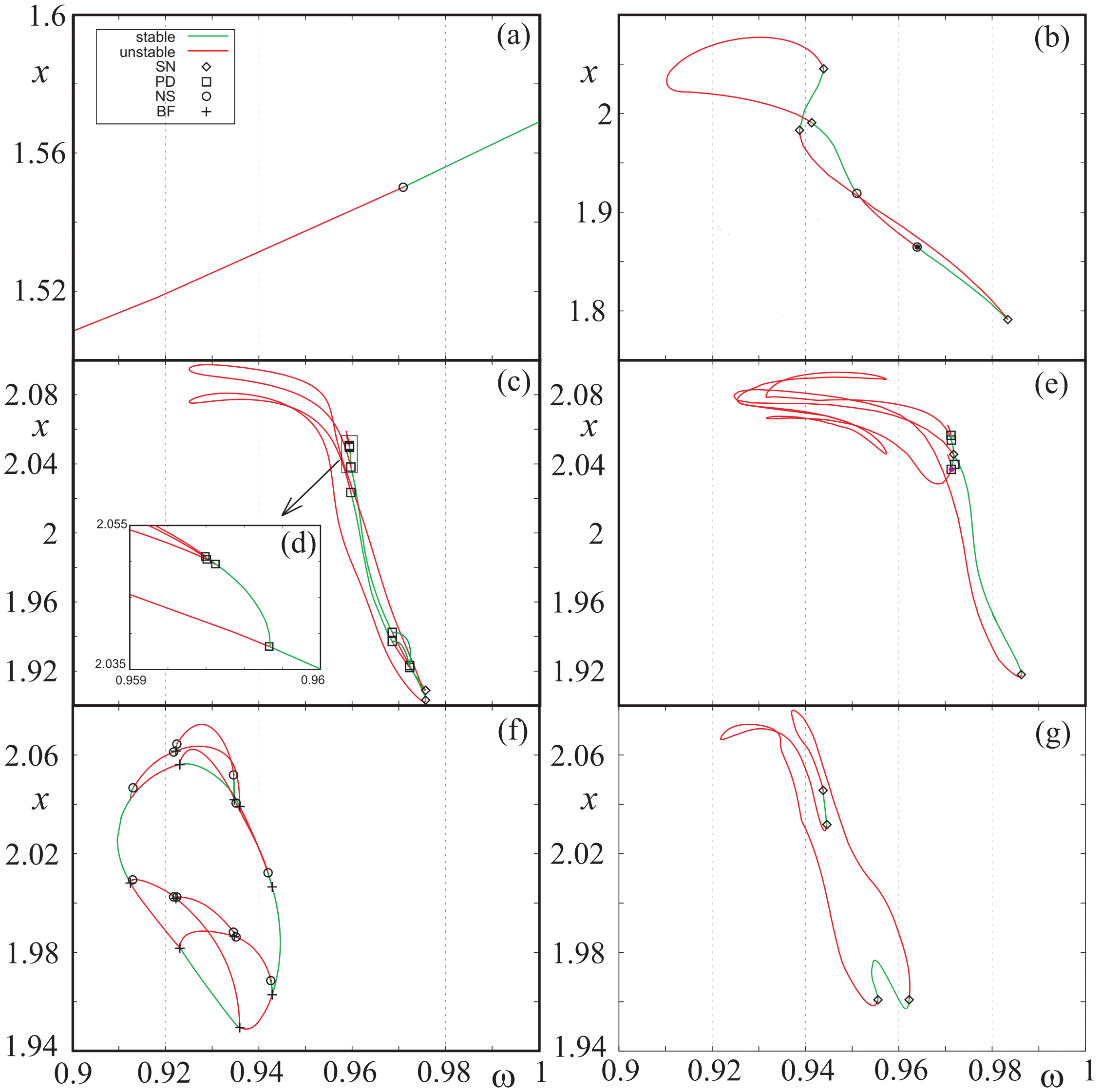

Fig. 2: Numerical continuation of the locked periodic solution: (a) $T-$PA, (b) $7T-$PA, (c) $9T-$PA, (d) zoom of PD scenario, (e) $11T-$PA, (f) $14T-$PA and (g) $21T-$PA. The green color corresponds to stable PS and red to unstable. The following bifurcations have been identify: SN (saddle-node), PD (period doubling) and NS (Neimark-Sacker) are marked by: diamond, square and circle respectively, the BF (branching point) is denoted by cross.

some of them can be considered as rare attractors. We identify two ways of transition to chaos (the PD scenario and the breakdown of the torus).

Typical examples of the BA of the attractors which exist in the multistable region are shown in Fig. 3(a-d). BA were computed using Dynamics II [Nusse and Yorke, 1998]. All the attractors are marked by black dots. In Fig. 3(a) BA of $\mathcal{A}_T$, $\mathcal{A}_{7T}$, two $\mathcal{A}_{9T}$ and $\mathcal{A}_{11T}$ are shown in green, yellow, blue, red and pink respectively. One can easily notice that the BA of the $\mathcal{A}_T$ (green one) is dominant for low initial values of x and $\dot{x}$. For large initial values one can see a competition between $\mathcal{A}_T$ and $\mathcal{A}_{11T}$. Two $\mathcal{A}_{9T}$ can be

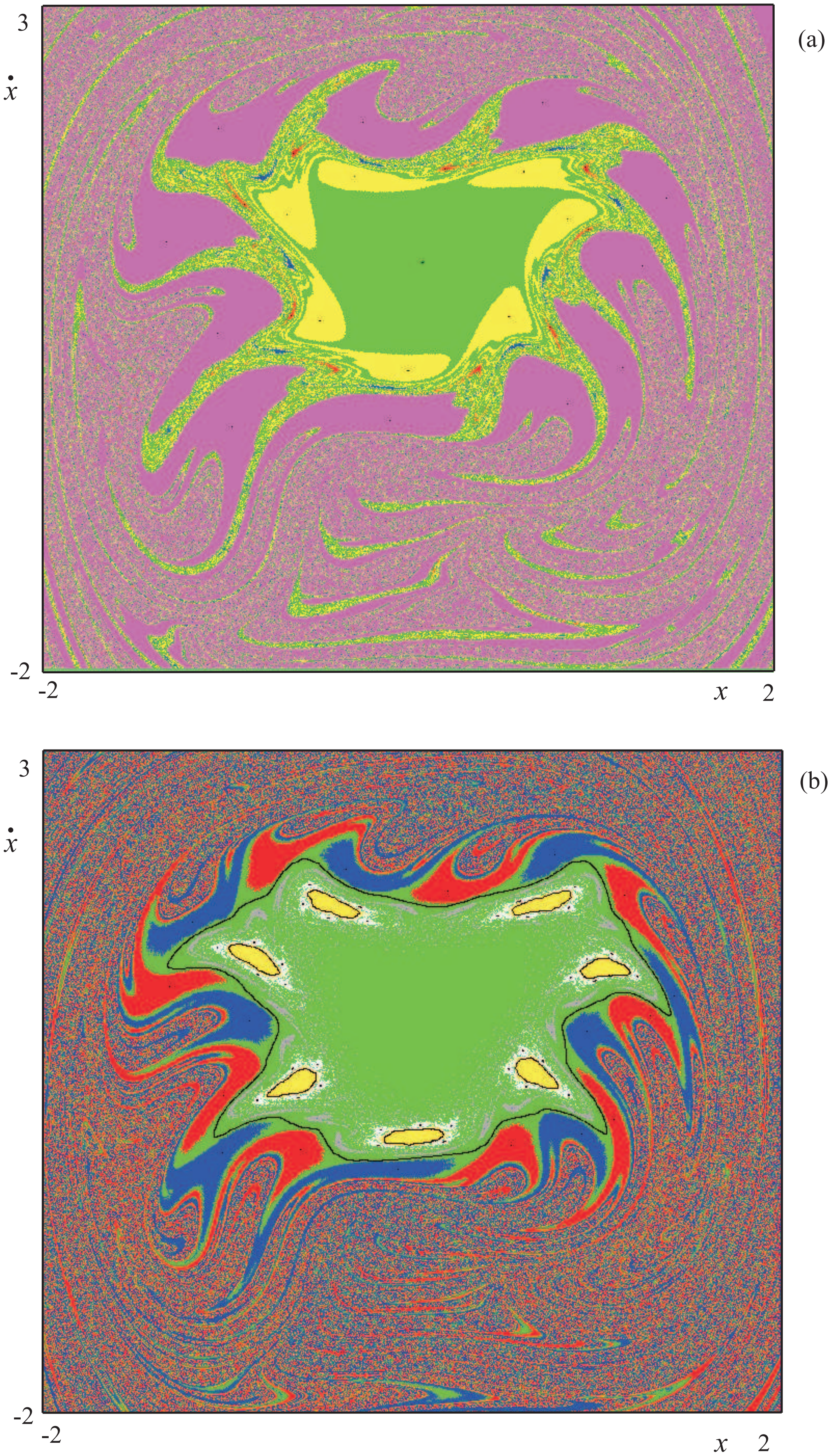

Fig. 3: The BA of system given by Eq. (2) for different frequency of excitation ω: (a) $\omega = 0.975$, (b) $\omega = 0.962$, (c) $\omega = 0.955$, (d) $\omega = 0.945$. All attractors are marked by black color, the BA of each attractor are shown with different colors (the detailed description is given in the text).

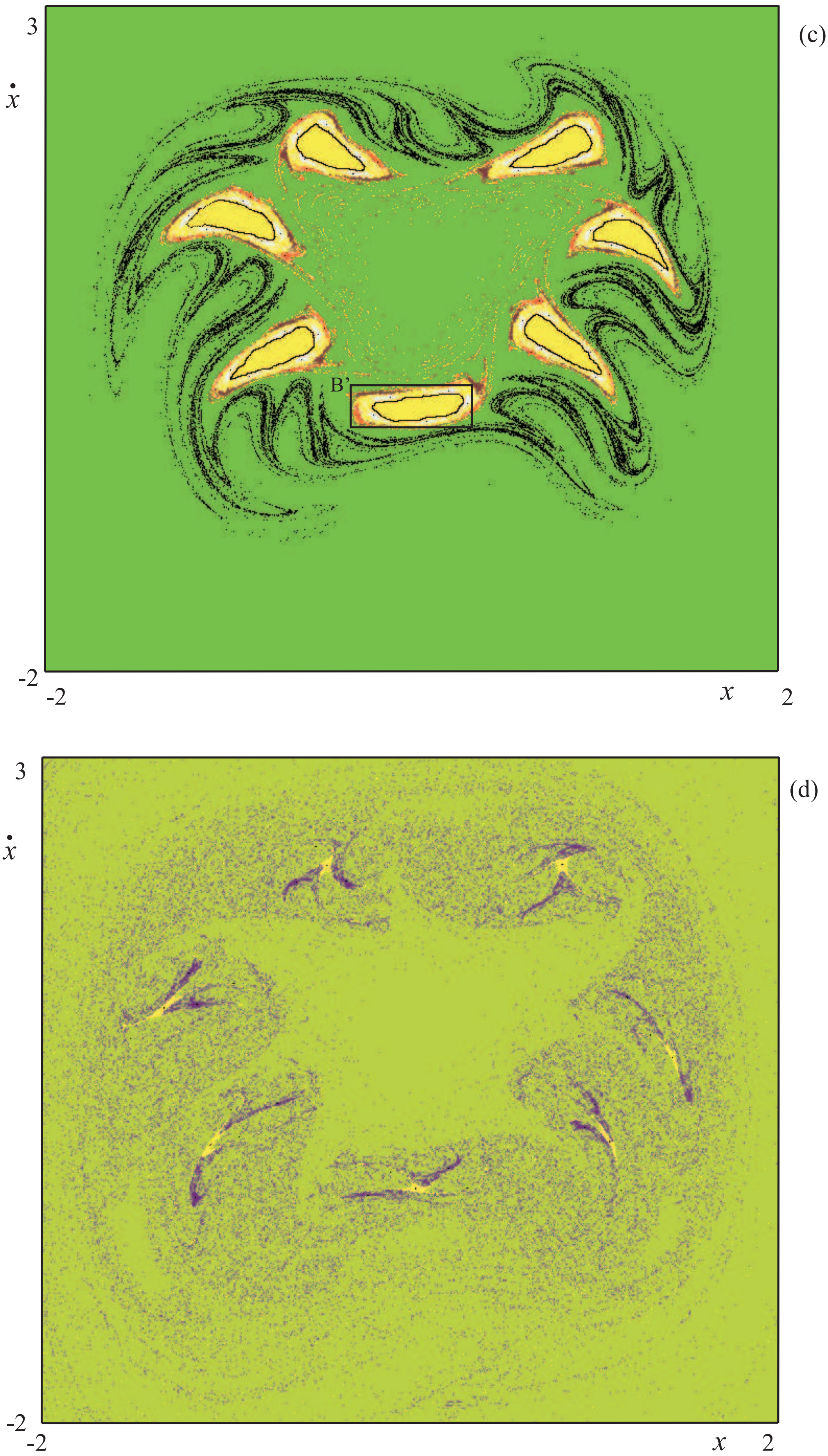

Fig. 3 (continued)

Type of attractor (color of BA)	$p(\mathcal{A})$ for $\mathcal{B}$	$p(\mathcal{A})$ for $\mathcal{B}'$
Chaotic $\mathcal{A}_{Ch}$ (green)	0.9227	0.0785
Quasiperiodic $\mathcal{A}_{Q7}$ (yellow)	0.0519	0.6982
Periodic $\mathcal{A}_{21T}$ (brown)	0.0087	0.0522
Periodic $\mathcal{A}_{35T}$ (white)	0.0086	0.1207
Periodic $\mathcal{A}_{21T}$ (orange)	0.0081	0.0504

Table 1: Probability of occurrence of the coexisting attractors for $\omega = 0.955$ (see Fig. 3(c)) and two possible initial conditions sets: $\mathcal{B} = [-2, 2] \times [-2, 3]$ and $\mathcal{B}' = [-0.34, 0.33] \times [-0.15, 0.17]$

considered as rare attractors.

At $\omega = 0.962$ (Fig. 3(b)) for both $T-$ and $7T-$ PS the NS bifurcations already occurred and one can observe toruses ($\mathcal{A}_Q$ and $\mathcal{A}_{Q7}$) with green and yellow BA respectively. Then, grey, white, blue and red BA correspond to $\mathcal{A}_{21T}$, $\mathcal{A}_{35T}$ and two $\mathcal{A}_{9T}$. The $\mathcal{A}_{21T}$ and $\mathcal{A}_{35T}$ are rare attractors.

Decreasing ω to 0.955 one can see the appearance of chaotic attractor with $\mathcal{A}_{Ch}$ BA, which is clearly dominant in the whole range of taken initial conditions. There is also a torus ($\mathcal{A}_Q$) with yellow BA and three rare attractors: two $\mathcal{A}_{21T}$ (brown and orange BA) and $\mathcal{A}_{35T}$ (white BA).

The last plot (Fig. 3(d)) corresponds to $\omega = 0.945$ where one can see just three BA: $\mathcal{A}_{7T}$ and two $\mathcal{A}_{14T}$ shown in yellow, violet and dark green respectively. The dominant one is $\mathcal{A}_{14T}$ with dark green BA.

The term rare attractors is strongly connected with the sets of accessible system parameters $\mathcal{C}$ and possible initial conditions $\mathcal{B}$. For example, consider the quasiperiodic attractor $\mathcal{A}_{7Q}$ (yellow BA in Fig. 3(c)). For simplicity we assume $\mathcal{C} = \mathcal{C}_{\mathcal{A}} = \omega = 0.955$. For this assumption and the set of possible initial conditions $\mathcal{B} = [-2, 2] \times [-2, 3]$ the probability that the system is on this quasiperiodic attractor is equal to $p(\mathcal{A}_{7Q}) = 0.0519$ and this attractor can be considered as the rare one. The dominant attractor is the chaotic attractor $p(\mathcal{A}_{Ch}) = 0.9227$ (green BA in Fig. 3(c)). If we consider the different set of possible initial conditions $\mathcal{B}' = [-0.34, 0.33] \times [-0.15, 0.17]$, this attractor has the highest probability of occurrence $p(\mathcal{A}_{7Q}) = 0.6982$. The second most probable one is periodic attractor $\mathcal{A}_{35T}$ (white BA in Fig. 3(c)) with $p(\mathcal{A}_{35T}) = 0.1207$, and previously dominant attractor $\mathcal{A}_{Ch}$ can be considered as the rare one with $p(\mathcal{A}_{Ch}) = 0.0785$. The probabilities of occurrence for all attractors for both cases of the sets of possible initial conditions are shown in Table 1.

To summarize, we present the bifurcational analysis of the phenomena leading to the creation of multistability in the externally excited van der Pol-Duffing oscillator in the neighborhood of the principal resonance. We give the evidence that the multistability for that system parameters where several different

Arnold's tongues overlap each other. In the considered case we identify the 1:1, 1:7, 1:9, 1:11 and 1:14, 1:15, 1:21 and 1:35 tongues overlap. In the multi-stable region some of the coexisting attractors have small BA and can be considered as the rare attractors. The probability of the system to evolve on one of these attractors is low but positive. In practice this means that the relatively small perturbation to the trajectory or fluctuation of the system parameters can result in the jump of the system to different, usually unexpected attractor. In many cases such a jump can be a disaster. We argue that rare attractors are typical for the forced nonlinear oscillators operating in the neighborhood of the principal resonance. The examples of such systems can be found in Blekham and Kuznetsova [2008].

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