Understanding Coin-Tossing

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It is commonly known that a toss of a fair coin is a random event and this statement is fundamental in the classical probability theory [1, 2, 3]. On the other hand, the dynamics of the tossed coin is described by deterministic equations, with no external source of random influence [4, 5, 6], so one can expect predictability in the results. It is possible to construct a mapping of the initial conditions (position, configuration, momentum, and angular momentum at the beginning of the coin motion) to a final observed configuration, that is, the coin terminates with its head (tail) side up or on its edge. The initial conditions which are mapped onto heads create a heads basin of attraction while those mapped onto tails create a tails basin of attraction [4]. The boundary which separates heads and tails basins consists of initial conditions mapped onto the coin standing on the edge. The structure of these boundaries has a significant impact on the problem of the coin-tossing predictability, that is, smooth basin boundaries allow predictability while fractal boundaries can lead to unpredictability [10, 11]. However, the precise structure of the heads-tails basin boundaries for a realistic model of a coin-tossing is unknown.

Here, we show that heads-tails basin boundaries are smooth, so the outcome of the coin-tossing is predictable. We have found that an increase in the number of impacts in the period when the coin bounces on the floor makes the basin boundaries more complex, and in the limiting case of an infinite number of impacts the behavior of the coin is chaotic and the basins of heads and tails become intermingled [12, 13, 14, 15, 16]. Our results demonstrate that although the coin-tossing is predictable, it can also approximate the random process and can serve as the foundation for understanding the behavior of physical (mechanical) randomizers [17, 18, 19]. We expect our results to be a new point in the discussion of the nature of random processes [17, 20].

The Coin Model

A coin can be modeled as a rigid body, namely a cylinder with a radius r and height h as shown in Figure 1. We consider a nonsymmetrical coin (the so-called cheat coin) for which the center of mass C is located at the distance $\xi_C \neq 0$, $\eta_C \neq 0$, $\zeta_C \neq 0$ from the geometrical center B. Any arbitrary position of a rigid body with respect to the fixed reference frame $Oxyz$ can be described by a combination of the position of the origin of the local reference frame $x'\ y'\ z'$ and the orientation (angular position) of this frame ξ, η, ζ [21, 22]. The local reference frame $x'\ y'\ z'$ is rigidly attached to the body and its axes are parallel to the xyz frame; ξ, η, ζ is the frame embedded and fixed in the body. For the origin of the local frames it is convenient to choose the geometric center of the body model B.

In our studies, we consider the following motion of the coin. We assume that the coin is thrown at the height z_0 with the initial conditions $\Phi_0 = \{x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, \psi_0, v_0, \phi_0, \omega_{\psi 0}, \omega_{v0}, \omega_{\phi 0}\}$, that is, the initial position of the center of mass is $\{x_0, y_0, z_0\}$, its initial velocity $\{\dot{x}_0, \dot{y}_0, \dot{z}_0\}$, the coin's initial orientation $\{\psi_0, v_0, \phi_0\}$, and the initial angular velocity $\{\omega_{\psi 0}, \omega_{v0}, \omega_{\phi 0}\}$. After a free fall when the z coordinate is zero, the coin collides with the parallel base (floor). It is assumed that at the collision, a portion of the coin energy is dissipated, that is, the collision is described by the restitution coefficient $\chi < 1$. The friction at the contact between the coin and the floor is described by the friction coefficient μ_{fr} [23]. After the collision, the coin's center of mass moves to height z_1 at which the total mechanical energy of the coin E is equal to its total energy in the moment after the collision E' minus the energy dissipated because of air resistance. Next, the coin moves on until it collides with the floor again. The calculations terminate when, after n-th collision, the total mechanical energy of the coin is smaller than the potential energy at the level of the coin's center of mass (approximately mgr, where g is the gravitational

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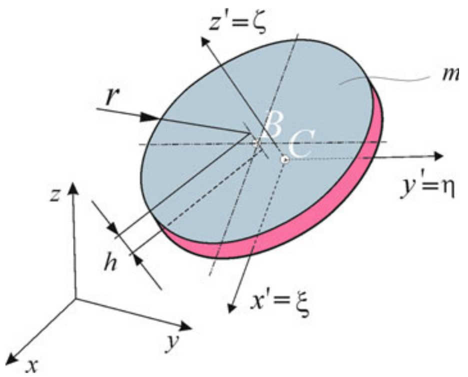

Figure 1. 3-dimensional model of the coin and its orientation in space.

acceleration), as this condition prevents the coin from flipping over. We also consider rotations of the coin on the floor. Full details of our model are given in [9].

The equations of motion describing the tossed coin are Newton's equations, with no external source of random influence, that is, fluctuations of air, thermodynamic or quantum fluctuations of the coin. These equations are discontinuous, so in analysing them one cannot apply continuity theorems or direct calculations of Lyapunov exponents. If the outcome of a long sequence of the coin-tossings is to give a random result, it can only be because the initial conditions vary sufficiently from toss to toss. The flow given by the equations of motion maps all possible initial conditions into one of the final configurations. The set of initial conditions which is mapped onto the heads configuration creates a heads basin of attraction $\beta(H)$ while the set of initial conditions mapped onto the tails configuration creates a tails basin of attraction $\beta(T)$. The boundary separating the heads and tails basins consists of initial conditions mapped onto the coin-standing-on-edge configuration [24]. For an infinitely thin coin, this set is a set of zero measure and thus with probability 1 the coin ends up either heads or tails. For a nonzero thinness of the coin this measure is not zero, but the probability that the edge configuration is stable is low.

Assume that one can set the initial conditions Φ_0 with uncertainty ϵ, where ϵ is small. If a ball $\mathcal{B}$ in the phase space centered at Φ_0 contains only the points which go to one of the final states, the outcome is predictable and repeatable. If in the ball $\mathcal{B}$ there are points leading to different final states (denote the set of points leading to heads as $\beta'(H)$ and the set points leading to tails as $\beta'(T)$), then the result of tossing is not predictable. One can calculate the probability of heads (tails) as $prob(heads) = \mu(\beta'(H))/\mu(\mathcal{B})$ ($prob(tails) = \mu(\beta'(T))/\mu(\mathcal{B})$), where μ is a measure of the sets $\beta'(H)$, $\beta'(T)$ and $\mathcal{B}$.

The possibility that heads-tails basin boundaries are fractal [10], riddled [12, 13, 14, 15], or intermingled [12, 16], is worth investigating. Near a given basin boundary, if the initial conditions are given with uncertainty ϵ, a fraction $f(\epsilon)$ of the initial conditions give an unpredictable outcome. In the limit $\epsilon \to 0$, $f(\epsilon) \propto \epsilon^{\alpha}$ where $\alpha < 1$ for fractal and $\alpha = 1$ for smooth boundaries. Fractal basins' boundaries are discontinuous (for example an uncountable sequence of disjoint stripes) or continuous (a snowflake structure) [11]. From the point of view of the predictability of the coin-toss the possibility of intermingled basins is the most interesting.

Let us briefly explain the meaning of the term intermingled basins of attraction. The basin $\beta(H)$ is said to be riddled by the basin $\beta(T)$ when it satisfies the following conditions: (i) it has a positive Lebesgue measure, (ii) for any point in $\beta(H)$, a ball in the phase space of arbitrarily small radius has a nonzero fraction of its volume in the basin $\beta(T)$. The basin $\beta(T)$ may or may not be riddled by the basin $\beta(H)$. If the basin $\beta(T)$ is also riddled by the basin $\beta(H)$, the basins are said to be intermingled. In this case, in any neighborhood of the initial condition leading to heads there are initial conditions which are mapped to tails, that is, there does not exist an open set of initial conditions which is mapped to one of the final states: an infinitely small inaccuracy in the initial conditions makes the state of the coin tossing unpredictable.

In our numerical calculations we consider the following coin data: $m = 20\ grams$, $r = 1.25\ cm$, $h = 0.2\ cm$ (former Polish 1 PLN coin made of a light aluminum-based alloy) and $\xi_C = 0.1\ cm$, $\eta_C = 0.1\ cm$, $\zeta_C = -0.02\ cm$. We considered the air resistance acting on the coin in both tangential and normal directions and described by the following coefficients $\lambda_n = 0.8$, $\lambda_\tau = 0.2$ [9]. The friction between the coin and the floor during the impact is described by friction coefficient $\mu_{fr} = 0.2$.

Results and Discussion

Figure 2 (a-d) shows the basins of attraction of heads and tails calculated for various coin models. The dark regions correspond to heads and the white ones to tails. The case of the coin terminating on the soft floor (restitution coefficient

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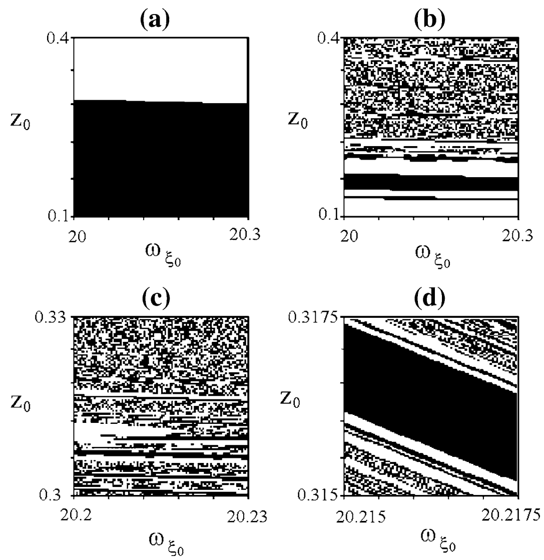

Figure 2. Basins of attraction of heads (black) and tails (white); (a) coin lands on the soft surface, (b-d) coin bounces on the floor, (c,d) enlargements of (b). The following parameters have been used: $x_0 = y_0 = 0, \dot{x}_0 = \dot{y}_0 = \dot{z}_0 = 0, \varphi_0 = \psi_0 = 0, \vartheta_0 = 7\pi/180$ rad, $\omega_{\zeta 0} = 0$, $\omega_{\eta 0} = 40.15$ rad/s.

$\chi = 0$) is shown in Figure 2(a). The case which allows the bouncing of the coin on the floor surface ($\chi = 0.6$) is shown in Figure 2(b). The structure of the basin boundaries for the case without bouncing on the floor is similar to the structure in the Keller model [7]. One can notice that the structure of the basin boundaries is more complex (looks fractal or intermingled) when the coin is allowed to bounce on the floor as can be seen in Figure 2(b). To check the possibility that these basins are fractal (or intermingled), the appropriate enlargements are presented in Figure 2(c,d). It can be seen that apart from the graininess because of the finite number of points, the boundaries are smooth (see Fig. 2d). Under further magnification no new structure can be resolved, that is, no evidence of intermingled or even fractal basin boundaries is visible. The same conclusion has been reached in the studies of simple one- or two-dimensional models [8, 4, 5]. Figure 2 (a-d) is based on the results obtained from numerically integrated equations of motion. We fixed all initial conditions except two, namely: the position of the coin mass center z_0 and the angular velocity $\omega_{\xi 0}$. We check that similar structures of the basin boundaries are observed when different initial conditions are allowed to vary. The two-dimensional sections of the phase space presented in Figure 2 (a-d) are a good indication of what happens in the entire phase space. We point out that the same structure of basins of attraction has been observed for the symmetrical coin [9].

This allows us to state our main result: for any initial condition Φ_0 there exists $\epsilon > 0$ such that a ball with radius ϵ centered at Φ_0 contains points which belong either to the set $\beta(H)$ or the set $\beta(T)$. In other words, if one can settle the initial condition with appropriate accuracy, the outcome of the coin-tossing procedure is predictable and repeatable.

Now we try to explain why for particularly small (but not infinitely small) ϵ the coin-tossing procedure can approximate a random process. A sequence of coin-tosses will be random if the uncertainty ϵ is large in comparison to the width W of the stripes characterizing the basins of attraction, so the condition $\epsilon >> W$ is essential for the outcome to be random [4]. It is interesting to note that the uncertainty ϵ depends on the mechanism of coin tossing while the quantity W is determined by the parameters of the coin.

In the case of the coin bouncing on the floor the structure of the heads and tails basin boundaries becomes complex (Figure 2b). In Figure 3(a-c) we show the calculations of these basins for a different number of impacts n. One can observe the face of the coin which is up after the n-th collision. Figure 3(a-c) shows the results for respectively 0, 3 and 10 collisions. With the increase of the collision numbers it is possible to observe that the complexity of the basin boundaries increases with the number of impacts. With the finite graininess (resolution) of Figure 3(a-c) these basin boundaries look fractal and one can speak about a fractalization-like process which can be observed with an increasing number of impacts.

To explain this process, consider the limiting case of an infinite number of impacts. Such a case neglects air resistance and assumes elastic impacts, that is, $\chi = 1$, and cannot be realized in a real experiment. Consider the map

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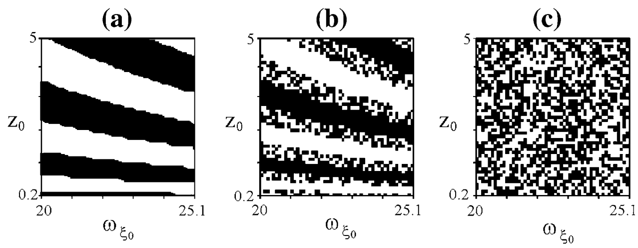

Figure 3. Basins of attraction indicating the face of the coin which is up after the n-th collision: (a) $n = 0$, (b) $n = 3$, (c) $n = 10$, heads and tails are indicated in black and white, respectively. The same parameters as in Figure 2 have been used.

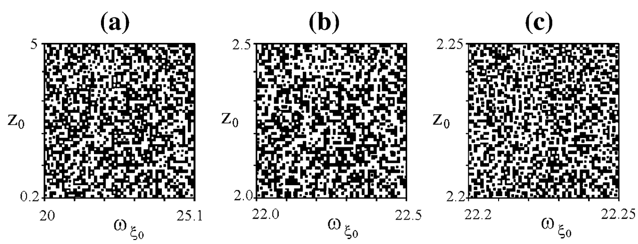

Figure 4. Basins of attraction of heads (black) and tails (white) in the case when the dissipation of energy is neglected; $n = 1000$ impacts, (b,c) are enlargements of (a). The same parameters as in Figure 2 have been used.

$U: [0, 2\pi] \rightarrow [0, 2\pi]$ which maps the point ϕ_n on the edge of the coin, which hits the floor at the n^{th} impact, to the point ϕ_{n+1} which hits the floor at the $(n+1)^{st}$ impact. Analysis of the time series of points $(\phi_1, \phi_2, \ldots)$ shows that the dynamics of U is chaotic when the largest Lyapunov exponent is positive (it has been numerically estimated from the time series to be 0.08). In this limiting case $(n \rightarrow \infty)$ the basins of heads and tails are intermingled and the outcome of the coin-tossing is unpredictable. Numerically, this can be observed when in the successive enlargements of the heads-tails basin boundaries the new structure is visible, as in Figure 4 (a-c) where the basins of heads and tails are calculated for $n = 1000$ impacts. The probability (we consider 10^6 different initial conditions) that a coin side which is up initially will still be up after 15 impacts is equal to 0.50987 and after 1000 impacts to 0.50006. This indicates that in the case of $n \rightarrow \infty$ this probability tends to 0.5.

In a real experiment, such a very large number of impacts cannot be realized because of the dissipation (inelastic impacts and air resistance) so the fractalization-like process has to stop. In our experiments [9] we observed that a typical coin falling from the height of 186 *cm* bounces on a wooden floor about 8–14 times. The existence of the chaotic process described by the map U introduces a time-sensitive dependence on initial conditions characterized by the positive maximum temporal Lyapunov exponent [25, 26, 27]. This sensitivity is responsible for the "fractalization" shown in Figure 3(a-c) and explains why the coins behave in practice as perfect randomizers.

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