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Synchronization of slowly rotating pendulums

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We study synchronization of a number of rotating pendulums mounted on a horizontal beam which can roll on the parallel surface. It has been shown that after the initial transient different states of pendulums' synchronization occur. We derive the analytical equations for the estimation of the phase differences between phase synchronized pendulums. After study of the basins of attraction of different synchronization states we argue that the observed phenomena are robust as they occur in the wide range of both initial conditions and system parameters.

Keywords: Rotating pendulum, motion complexity, synchronization, clusters, basins of attraction

1. Introduction

Over a few last decades the subject of synchronization has attracted the increasing attention from different fields [Pikovsky et al., 2001; Blekham, 1988; Pecora et al., 1997]. The study of synchronization of mechanical systems can be traced back to the works of the Dutch researcher Christian Huygens in XVIIth century (Huygens, 1665, 1673). He showed that a couple of mechanical clocks hanging from a common support were synchronized. Huygens found the pendulum clocks swung in exactly the same frequency and π out of phase, i.e., in antiphase synchronization. After the external perturbation, the antiphase state was restored within half an hour and remained indefinite. Recently, this idea has been rediscussed by a few groups of researchers who tested Huygens' idea [Pogromsky et al., 2003; Bennet et al., 2002; Senator, 2006; Dilao, 2009; Kumon et al., 2002; Fradkov & Andrievsky, 2007, J. Pantaleone, 2002; Ulrichs et al., 2009; Czolczynski et al., 2009(a,b)]. In all these works the possibility of the synchronization of the systems consisting of a number of oscillating pendulums has been investigated.

A pendulum is an archetype for strongly nonlinear dynamical systems, which naturally has been given a great deal of attention in literature [for example: Leven & Koch, 1981; 1985; Capecchi & Bishop, 1990; 1994;

Clifford & Bishop, 1995; El-Barki et al., 1999; Sudor & Bishop, 1999; Ge & Lin, 2000; Garira & Bishop, 2003; Grib et al. 2002; Szemplinska-Stupnicka et al., 2000; 2002; Yabuno et al., 2004; Mann & Koplov, 2006]. A pendulum can perform both oscillatory and rotational motion. Usually more attention has been paid to the oscillatory motion but recently more and more studies are concentrated on the rotational motion [Xua, 2005; Xu, et al., 2005b; 2007a; 2007b; Lenci & Rega, 2008; Lenci et al., 2008; Horton, 2008]. The rotational motions of the pendulum attracted more interest due to the concept of extracting energy from sea waves using pendulum dynamics proposed by Wiercigroch [2003]. Such a pendulum system can be used for converting of the pendulum base oscillations into the rotational motion of the pendulum mass (the oscillations of the base are caused by the sea waves, whereas the pendulum rotational motion provides the driving torque for an electrical generator).

Mechanical systems that contain rotating parts (for example vibro-exciters, unbalance rotors) are typical in engineering applications and for years have been the subject of intensive studies [Lee, (1993); Czolczynski, (1999); Vance et al. (2010)]. One problem of scientific interest, which among others occurs in such systems is the phenomenon of synchronization of different rotating parts [Blekhman, 1988; Balthazar et al., 2005 and references within]. Despite different initial conditions, after a sufficiently long transient, the rotating parts move in the same way - complete synchronization, or a permanent constant shift is established between their displacements, i.e., angles of rotation - phase synchronization [Boccaletti et al. 2002, Rosenblum et al. 1996; 1997]. Synchronization occurs due to dependence of the periods of rotating elements motion and the displacement of the base on which these elements are mounted [Fillipov et al., 1998].

In the previous papers [Czolczynski et al., 2009(a,b); 2011] we studied the synchronization problem for n identical and nonidentical pendulum clocks hanging from an elastically fixed horizontal beam. It was assumed that each pendulum performs a periodic motion which starts from different initial conditions. We showed that after a transient different types of synchronization between the pendulums can be observed. The first type is the in-phase complete synchronization in which all pendulums behave identically. In the second type one can identify the groups (clusters) of synchronized pendulums. We showed that only configurations of three and five clusters are possible and derive algebraic equations for the phase difference between the pendulums in different clusters. In the third type, which is possible only for even n, and identical clocks, one observes the anti-phase synchronization in $n/2$ pairs of pendulums.

In this paper we consider the dynamics of the similar system consisting of n pendulums mounted on the movable beam. The main difference is that the pendulums are excited by the external torques which are linearly dependent on the angular velocities of the pendulums. As the result of such excitation each pendulum rotates around its axis of rotation and can exhibit motion complexity. It has been shown that both complete and phase synchronizations of the rotating pendulums are possible. We derive the approximate analytical conditions for both types of synchronizations and equations which allow the estimation of the phase differences between the pendulums. Contrary to the case of oscillatory pendulums [Czolczynski et al., 2009(a,b); 2011] phase synchronization is not limited to three and five clusters configurations. We consider the case of slowly rotating pendulums and consider the influence of the gravity on their motion. Our results have been compared to those of [Blekham, 1988]. Differences of both analyses have been pointed out and explained. We give evidence that our results are robust as they exist in the wide range of system parameters.

The paper is organized as follows. In Sec.2 we describe the considered model of coupled rotating pendulums. Section 3 presents the analytical studies which allow to derive the synchronization condition. The examples of the configurations of the synchronized pendulums and its basins of attraction are given in Sec. 4. Finally, we summarize our results in Sec. 5.

2. The model

In the current studies we consider the system shown in Figure 1. The beam of mass M can move in the horizontal direction. The beam supports n rotating excited pendulums with the same length l and different masses m_i $(i{=}1,2,\dots,n)$. The rotation of the i-th pendulum is given by a variable φ_i and its motion is damped by the viscous friction described by damping coefficient c_φ. The forces of inertia, with which each pendulum acts on the beam, cause its motion in a horizontal direction (described by the coordinate x). The beam is considered as a rigid body so the elastic waves along it are not considered. We describe the phenomena which take place far below the resonances for both longitudinal and transverse oscillations of the beam. The beam is connected to a stationary base by the light spring with stiffness coefficient k_x and viscous damper with a damping coefficient c_x.. The pendulums are excited by external torques proportional to their velocities: $N_0 - \dot{\varphi}_i N_1$, where N_0 and N_1 are constant. If no other external forces act on the pendulum it rotates with the constant velocity $\omega = N_0/N_1$. If the system is in gravitational field (where g=9.81 is the acceleration due to gravity), the weight of the pendulum causes the unevenness of its rotation, i.e., the pendulum slows down when the center of its mass goes up and accelerates when the center of its mass goes down.

The system equations can be written in a form of Euler-Lagrange equations:

$$m_i l^2 \ddot{\varphi}_i + m_i \ddot{x} l \cos\varphi_i + c_\varphi \dot{\varphi}_i + m_i g l \sin\varphi_i = N_0 - \dot{\varphi}_i N_1 \quad (1)$$

$$\left(M + \sum_{i=1}^{n} m_i\right) \ddot{x} + c_x \dot{x} + k_x x = \sum_{i=1}^{n} m_i l \left(-\ddot{\varphi}_i \cos\varphi_i + \dot{\varphi}_i^2 \sin\varphi_i\right). \quad (2)$$

In our numerical simulations eqs (1-2) have been integrated using Runge-Kutta method. Different types of pendulums synchronization states have been identified. To study the stability of such a solution of eqs (1-2) we add perturbations δ_i and σ to the variables φ_i and x and obtain the following linearized variational equation:

$$m_i l^2 \ddot{\delta}_i + m_i \ddot{\sigma} l \cos\varphi_i + m_i l \delta_i \left(g \cos\varphi_i - \ddot{x} \sin\varphi_i\right) + c_\varphi \dot{\varphi}_i = 0 \quad (3)$$

$$M + \sum_{i=1}^{n} m_i \ddot{\sigma}_i + \sum_{i=1}^{n} \left(m_i l \delta_i \cos\varphi_i - m_i l \dot{\varphi}_i^2 \delta_i \cos\varphi_i - m_i l \ddot{\varphi}_i^2 \sin\varphi_i - 2 m_i l \dot{\varphi}_i \dot{\delta}_i \sin\varphi_i\right) + c_x \dot{\sigma} + k_x \sigma = 0 \quad (4)$$

The solution of eqs (1-2) given by $\varphi_i(t)$ and $x(t)$ is stable when the solution of eqs (3-4) δ_i and σ tend to zero for $t \to \infty$. All the pendulums synchronization states described in this paper fulfils this relation.

In [Blekhman, 1988] it has been has shown that the oscillation frequency α_x of the oscillator consisting of mass U (the mass of the beam plus the masses of pendulums mounted to it) suspended on the spring k_x and damper c_x plays an important role in the study of the stability of the synchronized states. The equation describing this oscillator has the following form (compare eq.(2))

$$U\ddot{x} + c_x \dot{x} + k_x x = 0. \quad (5)$$

For the small value of the damping coefficient c_x we can assume that

$$\alpha_x = \sqrt{\frac{k_x}{U}}. \quad (6)$$

In the calculations presented in this paper, c_x has been assumed as sufficiently small (equivalent to the arbitrarily chosen value of the logarithmic decrement of damping Δ=ln(1.5)). As such a damping does not significantly alter the period of free oscillations, c_x can be expressed as follows:

$$c_x = \frac{\Delta\sqrt{k_x U}}{\pi}. \tag{7}$$

The introduction of α_x allows the comparison of our results with that of [Blekhman, 1988].

3. Condition for synchronization

In our studies we assume that the fluctuations of the pendulums' angular velocities $\dot{\varphi}_i$ which are caused by the motion in the gravitational field, can be described by the harmonic functions with frequency ω. Under these assumptions one gets the following functions:

$$\begin{aligned} \dot{\varphi}_i &= \omega + \eta\cos\left(\omega t + \beta_i\right), \\ \varphi_i &= \omega t + \beta_i + \tfrac{\eta}{\omega}\sin\left(\omega t + \beta_i\right), \\ \ddot{\varphi}_i &= -\eta\omega\sin\left(\omega t + \beta_i\right). \end{aligned} \tag{8}$$

which determine respectively the angular velocity, angle of rotation and angular acceleration of the i-th pendulum. Notice that in the case of high speed systems like rotors these fluctuations are negligible [Lee, 1993; Czolczynski, 1999; Vance et al. 2010].

Right hand side of eq.(2) describes the total force with which the n pendulums act on the beam M:

$$F = \sum_{i=1}^{n} m_i l\left(-\ddot{\varphi}_i\cos\varphi_i + \dot{\varphi}_i^2\sin\varphi_i\right). \tag{9}$$

Substituting eq.(8) into eq.(9) one gets:

$$F = \sum_{i=1}^{n}\left(m_i l\begin{pmatrix} \eta\omega\sin(\omega t + \beta_i)\cos\left(\omega t + \beta_i\right) + (\eta/\omega)\sin(\omega t + \beta_i) + \\ +\left(\omega + \eta\cos(\omega t + \beta_i)\right)^2\sin\left(\omega t + \beta_i\right) + (\eta/\omega)\sin(\omega t + \beta_i) \end{pmatrix}\right). \tag{10}$$

Assuming that the value of(η/ω) is small one gets

$$\begin{aligned} \cos\left(\tfrac{\eta}{\omega}\sin(\omega t + \beta_i)\right) &= 1.0, \\ \sin\left(\tfrac{\eta}{\omega}\sin(\omega t + \beta_i)\right) &= \tfrac{\eta}{\omega}\sin(\omega t + \beta_i) \end{aligned}$$

and eq.(10) can be rewritten as follows

$$F = \sum_{j=1}^{4} f_j\left(\sum_{i=1}^{n}\left(m_i l\sin(j\omega t + j\beta_i)\right)\right) = \sum_{j=1}^{4} f_j\left(\sin j\omega t\sum_{i=1}^{n}\left(m_i l\cos j\beta_i\right) + \cos j\omega t\sum_{i=1}^{n}\left(m_i l\sin j\beta_i\right)\right), \tag{11}$$

where:

$$f_1 = \omega^2,\ f_2 = 2\eta\omega + \frac{\eta^3}{4\omega},\ f_3 = \eta^2,\ f_4 = \frac{\eta^3}{8\omega}. \tag{12}$$

Eq.(11) shows that the resultant force of inertia F which operates on the beam M consists of four harmonics. Notice that this result is qualitatively different from the one obtained for a system of oscillating pendulums [Czolczynski et al., 2009a; 2009b] where this force consists of the first and third harmonics only.

Eqs (12) indicate that the second, third and fourth harmonics of force F exist only when the gravitational field causes the uneven motion of the pendulums, i.e., the instantaneous angular velocities fluctuate around the average values. When the pendulums rotate in the parallel plane, or when one can neglect the influence of the gravity due to the high speed of the rotation there is only one harmonic component of the force F.

Substituting eq.(11) into eq.(2) one gets

$$U\ddot{x} + c_x\dot{x} + k_x x = \sum_{j=1}^{4} f_j \left(\sum_{i=1}^{n} (m_i l \sin(j\omega t + j\beta_i)) \right), \tag{13}$$

where

$$U = M + \sum_{i=1}^{n} m_i.$$

Assuming that c_x is small the motion of the beam M can be described as follows

$$\begin{aligned} x &= \sum_{j=1}^{4} \frac{f_j}{k_x - j^2\omega^2 U} \left(\sum_{i=1}^{n} (m_i l \sin(j\omega t + j\beta_i)) \right), \\ \ddot{x} &= \sum_{j=1}^{4} \frac{-j^2\omega^2 f_j}{k_x - j^2\omega^2 U} \left(\sum_{i=1}^{n} (m_i l \sin(j\omega t + j\beta_i)) \right). \end{aligned} \tag{14}$$

In the equation of k-th pendulum motion (1) one can identify the component

$$M_{Sk} = m_k l \ddot{x} \cos\varphi_k, \tag{15}$$

which can be called a synchronization momentum. The index k ($k = 1, 2, \ldots, n$) has been introduced to allow description of the influence of remaining $n-1$ pendulums on the k-th one.

This is the torque with which the beam M acts on the k-th pendulum. The work done by this increases or decreases the energy accumulated in the k-th pendulum In the synchronization state of the oscillations of the beam M are periodic. The work W_{Sk} done by the torque M_{Sk} during one period of these oscillations is equal to zero:

$$W_{Sk} = \int_0^T (m_i l \ddot{x} \cos\varphi_i)\dot{\varphi}_i dt = 0. \tag{16}$$

Substituting eqs (8) and (14) into eq.(16) and denoting

$$A_j = \frac{-j^2\omega^2 f_j}{k_x - j^2\omega^2 U}$$

one gets

$$W_{Sk} = m_k l \int_0^T \left[\sum_{j=1}^{4} A_i \left(\sum_{i=1}^{n} (m_i l \sin(j\omega t + j\beta_i)) \right) \right] \cos\left(\omega t + \beta_k + \frac{\eta}{\omega} \sin(\omega t + \beta_k) \right) (\omega + \eta \cos(\omega t + \beta_k))\, dt = 0. \tag{17}$$

Assuming that η is significantly smaller than ω, eq. (17) can be rewritten as

$$W_{Sk} = m_k l \int_0^T \left[\sum_{j=1}^{4} A_i \left(\sum_{i=1}^{n} (m_i l \sin(j\omega t + j\beta_j)) \right) \right] \left(\cos(\omega t + \beta_k) - \frac{\eta}{\omega} \sin^2(\omega t + \beta_k) \right) (\omega + \eta \cos(\omega t + \beta_k))\, dt = 0. \tag{18}$$

Omitting the components with η and η^2 as small (this is consistent with the proceedings in the analysis based on the small parameter method [Blekham, 1988], after transformations we obtain

$$W_{Sk} = m_k l^2 \pi A_1 \left(\cos\beta_i \sum_{i=1}^{n} m_i \sin\beta_i - \sin\beta_i \sum_{i=1}^{n} m_i \cos\beta_i \right) = m_k l^2 \pi A_1 \sum_{i=1}^{n} m_i \sin(\beta_i - \beta_k) = 0. \tag{19}$$

From eq.(19) one can calculate the value of phase angles β_i in the synchronized state. Eq.(19) is fulfilled when

$$\beta_1 = \beta_2 = ... = \beta_n \tag{20}$$

(the case of complete synchronization) or when

$$\sum_{i=1}^{n} m_i \cos\beta_i = 0, \ \sum_{i=1}^{n} m_i \sin\beta_i = 0. \tag{21}$$

(the case of phase synchronization).

These results (eqs (20) and (21)) are identical to these obtained by the method of small parameter [Blekhman, 1988].

In the system with identical pendulums, the condition (21) has the form (we assume $\beta_1 = 0$)

$$\begin{aligned} &1.0 + \cos\beta_2 + \cos\beta_3 + ... + \cos\beta_n = 0, \\ &\sin\beta_2 + \sin\beta_3 + ... + \sin\beta_n = 0. \end{aligned} \tag{22}$$

It can be shown that eq.(22) is satisfied by the phase angles defined by the formula

$$\beta_i = \frac{2\pi(i-1)}{n}. \tag{23}$$

For example eq. (23) shows that for n=3, phase differences are respectively equal β_1=0^o, β_2=120^o, β_3=240^o, and for n=2, β_1=0^o, β_2=180^o.

4. Examples

4.1. *Two pendulums*

(n=2)

In our numerical simulations we consider the system (1-2) with the following parameter values: $m_1 = m_2 = 1.00$, $l = 0.25$, $c_\varphi = 0.01$, $N_0 = 5.00$, $N_1 = 0.50$, $M = 6.00$. One can calculate that $\omega = 10.0$

and $U = 8.0$. We consider different values of stiffness coefficient k_x of the spring connecting the beam M with a fixed foundation so the beam can oscillate above or below the resonance, i.e., the frequency α_x is respectively smaller or larger than the angular velocity ω. Damping coefficient c_x is given by eq. (7).

Typical time series of pendulums' velocities and displacements are shown in Figure 2(a,b). The unit of time on the horizontal axis is the number $K = \omega t/2\pi$ i.e., the number of complete revolutions of the pendulum rotating with constant angular velocity ω. Figure 2(a) shows the angular velocities of pendulums $\dot{\varphi}_1$and $\dot{\varphi}_2$ for a system with low stiffness coefficient k_x=100.0, so $\alpha_x = (100.0/8.0)^{0.5} = 3.53 < 10.0 = \omega$. The following initial conditions have been considered: φ_{10}=0.0^o, φ_{20}=45.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = 0.0$. As one can see, after the transient the phase difference between the pendulums' velocities tends to 180^o. Figure 2(b) shows the angular displacement of pendulum 2, $\varphi_2 - \varphi_1$, related to the displacement of the first pendulum. One can notice that after the decay of the transient, this angle oscillates around a constant average value 180^o and the system reaches the state of antiphase synchronization. This state for small values of k_x is reachable for any initial conditions. Shown in both figures the fluctuations of angular velocities and displacements, are caused by the weight of pendulums, i.e., pendulum speeds during the motion down and slows when its mass rise up.

The pendulums' configurations characteristic for the system (1-2) with n=2 pendulums and its basins of attraction are shown in Figure 3(a-d). Figure 3(a) presents the configuration of antiphase synchronization with $\beta_1 = 0.0^o$ and $\beta_2 = 180.0^o$. Notice that the same values of β_1 and β_2 can be calculated analytically from eq.(22) and condition (21) is fulfilled. The configuration complete synchronization is presented in Figure 3(b). This configuration is observed for larger values of coefficient k_x when condition (20) is fulfilled. Figure 3(c) shows the basins of attraction of the complete (white color) and anti-phase (red color) synchronization states in the system with a stiffness coefficient of $k_x = 3600.0$. The basins are shown in the $\varphi_{10} - \varphi_{20}$ plane with fixed initial velocities $\dot{\varphi}_{10} = \dot{\varphi}_{20} = 0.0$. These basins for systems with different values of the stiffness coefficient k_x, shown on the plane $k_x - \varphi_{20}$ (φ_{10}=0.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = 0.0$) are presented in Figure 3(d). The results of Figure 3(d) and predictions of [Blekhman, 1988] are significantly different. Blekhman [1988] predicts the existence of complete (for $k_{x\omega} = U\omega^2 < 800.0$) and antiphase (for $k_{x\omega} = U\omega^2 > 800.0$) independently of initial conditions. Meanwhile, Figure 3(d) shows, that the boundary between the basins of attraction of complete and antiphase synchronizations is located significantly below the value of $k_{x\omega}$

and is not horizontal. For $k_x < k_{x1} \approx 400.0$, independently of initial conditions one observes the antiphase synchronization while for $k_{x1} < k_x < k_{x2} \approx 600.0$ there exists the coexistence of complete and antiphase synchronizations. In the interval $k_{x2} < k_x < k_{x3} \approx 1840.0$ independently of initial conditions the system (1-2) reaches the state of complete synchronization and for larger values of $k_x > k_{x3}$ we have the coexistence of both synchronization states again.

4.2. *Three pendulums (n=3)*

Let us consider the system (1-2) with the following parameter values: $m_1 = m_2 = m_3 = 1.00$, $l = 0.25$, $c_\varphi = 0.01$, $N_0 = 5.00$, $N_1 = 0.50$, $M = 6.00$. As in Sec. 4.1 one can calculate that $\omega = 10.0$ and $U = 9.0$ (due to $n = 3$). The values of stiffness and damping coefficients k_x and c_x have been taken as in Sec. 4.1.

Typical time series of pendulums' velocities and displacements in the case of phase synchronization are shown in Figure 4(a,b). Figure 4(a) shows the angular velocities of pendulums $\dot{\varphi}_1$, $\dot{\varphi}_2$ and $\dot{\varphi}_3$ for a system with low stiffness coefficient $k_x = 100.0$ and $\alpha_x = (100.0/9.0)^{0.5} = 3.33 < 10.0 = \omega$. The following initial conditions have been considered: $\varphi_{10} = 0.0^o$, $\varphi_{20} = 45.0^o$, $\varphi_{30} = 90.0^o$, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$. As in the previous plots (Figure 2(a,b)) the unit of time on the horizontal axis is the number $K = \omega t/2\pi$ i.e., the number of complete revolutions of the pendulum rotating with constant angular velocity ω. As one can see, after the decay of transients the phase difference between the pendulums velocities tends to the constant value of 120^o. Figure 4(b) shows the angular displacement of pendulums, $\varphi_2 - \varphi_1$ and $\varphi_3 - \varphi_1$, related to the displacement of the first pendulum. One can notice that these angles in what follows referred as the relative displacements oscillate around a constant average values 120^o and 240^o. Such a state of phase synchronization is obtained for k_x=100.0 and arbitrary initial conditions. Shown in both figures the angular velocity fluctuations and movements in relative terms, are caused by the motion in gravitational field. Numerically estimated phase shifts $\beta_2 = 120.0^o$ and $\beta_3 = 240.0^o$ are in good agreement with the values calculated analytically from eqs.(19).

In another example, it is assumed that $k_x = 3600.0$, so $\alpha_x = (3600.0/9.0)^{0.5} = 20.0 > 10.0 = \omega$ and the system of the beam and three pendulums eqs. (1-2) is below the resonance. We consider the following initial conditions: $\varphi_{10} = 0.0^o$, $\varphi_{20} = 45.0^o$, $\varphi_{30} = 90.0^o$, Figure 5(a,b) shows that after a transitional period the angular velocities of all three pendulums are the same and the relative displacements $\varphi_2 - \varphi_1$ and $\varphi_3 - \varphi_1$ tend to zero, so one observes the state of complete synchronization. Due to the existence of gravitational field we observe the fluctuations of the pendulums' motion caused by their weights.

In the system (1-2) with a stiffness coefficient of k_x=3600.0 and different initial conditions (for example φ_{10}=0.0^o, φ_{20}=90.0^o, φ_{30}=180.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$ one observes a different type of synchronization as shown in Figure 6(a,b). After a transitional period angular velocities$\dot{\varphi}_1$and$\dot{\varphi}_3$ tend to each other and are different than $\dot{\varphi}_2$; relative displacement $\varphi_3 - \varphi_1$ reaches a constant value 360.0^o, so $\varphi_3 = \varphi_1$, and $\varphi_2 - \varphi_1$=180.0^o. Two pendulums 1 and 3 create a cluster which is in anti-phase with pendulum 2.

Our numerical results show that in the system (1-2) with n=3 pendulums three different configurations of synchronized pendulums are possible, as shown in Figure 7(a-c). Figure 7(a) presents the phase synchronization with phase shifts between pendulums: β_1=0.0^o, β_2=120.0^o i β_3=240.0^o(condition (21) is fulfilled) which exists for sufficiently small values of k_x <370 (regardless of initial conditions). Complete synchronization ($\beta_1 = \beta_2 = \beta_3$and condition (20) is fulfilled) which exists for the appropriate values of k_x (370< k_x <1880) regardless of initial conditions and which for sufficiently large values of k_x (k_x >1880) coexists with antiphase synchronization is described in Figure 7(b). In contrast to the previously studied systems with oscillating pendulums [Czolczynski et al., 2009(a,b)] one can observe the phenomenon of antiphase synchronization of a single pendulum with the cluster of two other pendulums. Figure 7(c) presents the anti-phase synchronization β_1=0.0^o (or β_2=0.0^o or β_3=0.0^o) and two other phase shift angles equal to 180.0^o(condition (21) is fulfilled). This configuration co-exists with a complete synchronization for sufficiently large values of k_x (k_x >1880). Depending on initial conditions the cluster is created of pendulums 1-2, 1-3 or 2-3.

The basins of attraction of different states of pendulums' synchronization are shown in Figure 8(a,b). Figure 8(a) shows the basins of attraction of complete (white color) and anti-phase (brown, green, pink colors for different pairs of pendulums in the cluster) synchronization states for a system with stiffness coefficient k_x=3600.0. The basins are shown in the $\varphi_{20} - \varphi_{30}$ plane (φ_{10}=0.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$). Figure 8(b) shows the basins of attraction of complete (white color), anti-phase (brown, green, pink colors for different pairs of pendulums in the cluster) and phase (red color at the bottom) for different values of stiffness coefficient k_x. The following initial conditions have been considered: φ_{10}=0.0^o, φ_{20}=180.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$.

The results obtained by the numerical integration of equations of motion (1,2) are significantly different from the results obtained by the method of small parameter [Blekhman, 1988]. For example [Blekhman, 1988] predicts the existence of complete and phase synchronization (the second one with the same phase shifts as in our studies, i.e., 120^o and 240^o) . It has been stated that for the systems with stiffness coefficient $k_{x\omega} = U\omega^2 < 900.0$ independently of initial conditions the phase synchronization occurs while for larger values of $k_{x\omega}$ the complete synchronization takes place. Contrary to this statement Figure 8 shows that the boundary between the basins of attraction phase and complete synchronization takes place at the level k_x=370.0 (almost three times lower).

Another significant difference between our results and these of [Blekhman, 1988] is the existence of anti-face synchronization of a single pendulum and a cluster consisting of two pendulums ([Blekhman, 1988] does not prescribe such configuration). For k_x >1880.0 this configuration co-exits with a complete synchronization of all pendulums. Notice that the method of small parameter used in [Blekhman, 1988] does not allow the identification of the coexisting configurations.

4.3. *Large systems*

We studied the systems with up to 100 rotating pendulums. It has been found that for larger $n$same types of synchronization are observed. Their examples are shown in Figures 9-11. Figure 9(a,b) presents the phase synchronization of n=20 pendulums in the system (1-2) with k_x=1000.0, M=20.0, and the following initial conditions:$\varphi_{i0} = \frac{i\times 90^o}{20}$, $\dot{\varphi}_{i0} = 0.0$. Figure 9(a) shows that pendulums' velocities $\dot{\varphi}_1 \div \dot{\varphi}_{20}$ oscillate around the average value close to ω. Angular displacements $\varphi_i - \varphi_1$ tend to the constant values which differ by 18^oas can be seen in Figure 9(b). The complete synchronization of 20 pendulums is described in Figure 10(a,b). We consider the system (1-2) with k_x =20000.0, M=20.0 and initial conditions: $\varphi_{i0} = \frac{i\times 140^o}{20}$, $\dot{\varphi}_{i0} = 0.0$. The velocities of all pendulums $\dot{\varphi}_1 \div \dot{\varphi}_{20}$ oscillate around the constant average value ω (Figure 10(a)) and angular displacements $\varphi_i - \varphi_1$ tend to zero, i.e., the displacements of all pendula are the same (Figure 10(b)). The example of the synchronization in clusters is presented in Figure 11(a,b). We consider the same system as in the previous example with the following initial conditions: $\varphi_{i0} = \frac{i\times 180^o}{20}$, $\dot{\varphi}_{i0} = 0.0$. Figure 11(a) shows that pendulums' velocities $\dot{\varphi}_1 \div \dot{\varphi}_{20}$ oscillate around the average value close to ω. The angular displacements $\varphi_i - \varphi_1$ tend to two constant values 0^o or 180^o as can be seen in Figure 11(b). Two clusters of synchronized pendulums have been created (they consist of 7 and 13 pendulums). The clusters are synchronized in antiphase.

Contrary to the case of oscillating pendulums [Czolczynski et al., 2009(a, b)] rotating pendulums are not grouped in three or five clusters only. The lack of this restriction causes that in the system (1,2) depending on initial condition one can observe a great variety of different clusters' configurations. The number of configurations grows with a number of pendulums n.

5. Conclusions

In the considered system consisting of the horizontally moving beam on which externally excited rotating pendulums are mounted one can observe both complete and phase synchronization of the pendulums. In the state of phase synchronization, the average phase shifts between the pendulums are constant. As a result of constant acceleration and deceleration of the pendulums due to the gravity, the instantaneous phase shifts fluctuate around these averages. Similarly to the case of oscillating pendulums [Czolczynski et al., 2009(a, b)], we observe the creation of the clusters of completely synchronized pendulums. Contrary to the oscillating case the rotating pendulums are not grouped in three or five clusters only and a motion complexity is higher.

For some intervals of the system parameters we found the motion complexity with the co-existence of the different stable synchronized states as is clearly visible when one calculate basins of attraction. Such a coexistence has not been reported previous in the context of rotating pendulums. It has not been identified in Blekhman [1988], as the small parameter methods used there, do not allow it.

Our approximate analytical studies allow the derivation of the conditions for both types of synchronization as well as the equation for estimation of the shifts between the phases of the synchronized pendulums. Both results are in good agreement with numerical results.

Finally, we give evidence that all synchronization states reported in this paper are robust as they can be observed on the wide intervals of the system parameters.

Acknowledgment

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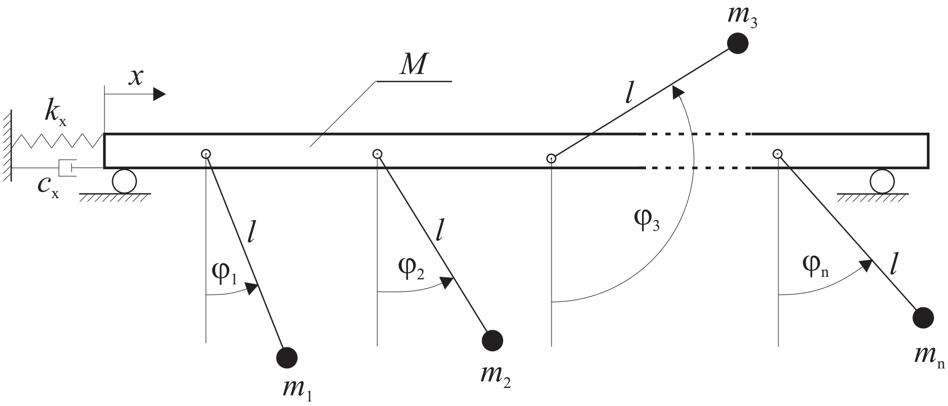

Fig. 1: The model of the considered system.

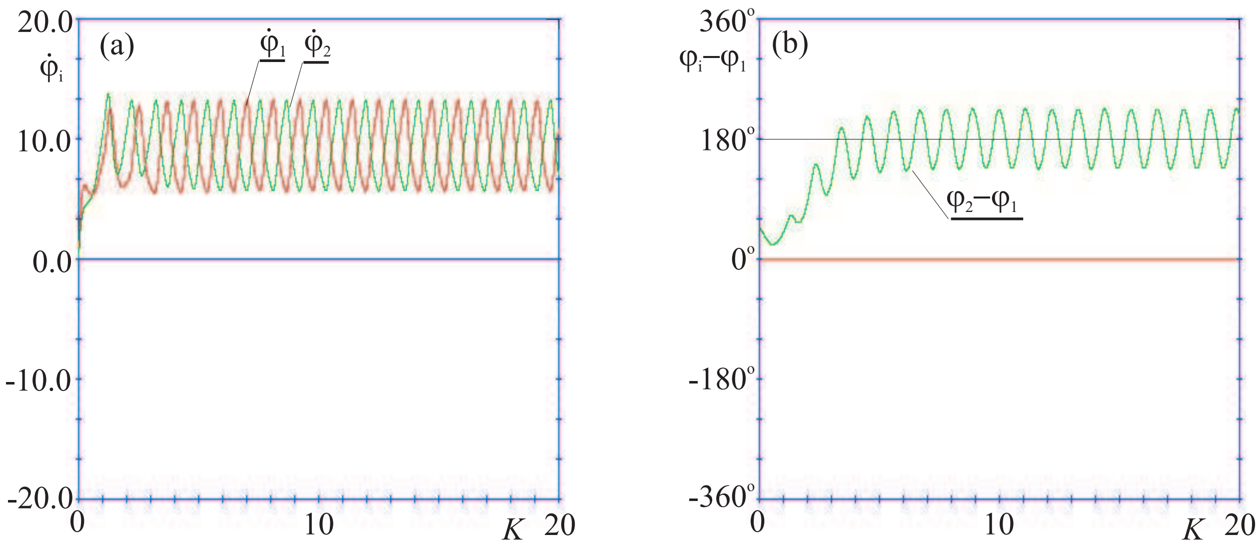

Fig. 2: Time series of pendulums' velocities and displacements calculated from eqs (1,2): $m_1 = m_2 = 1.00$, $l = 0.25$, $c_\varphi = 0.01$, $N_0 = 5.00$, $N_1 = 0.50$, $M = 6.00$, $\omega = 10.0$, $U = 8.0$ (the unit of time on the horizontal axis is the number $K = \frac{\omega t}{2\pi}$ i.e., the number of complete revolutions of the pendulum rotating with constant angular velocity ω); (a) angular velocities $\dot{\varphi}_1$ and $\dot{\varphi}_2$ for a system (1,2) with low stiffness coefficient k_x=100.0, φ_{10}=0.0^o, φ_{20}=45.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = 0.0$; (b) angular displacement of pendulum 2: $\varphi_2 - \varphi_1$ related to the displacement of the first pendulum.

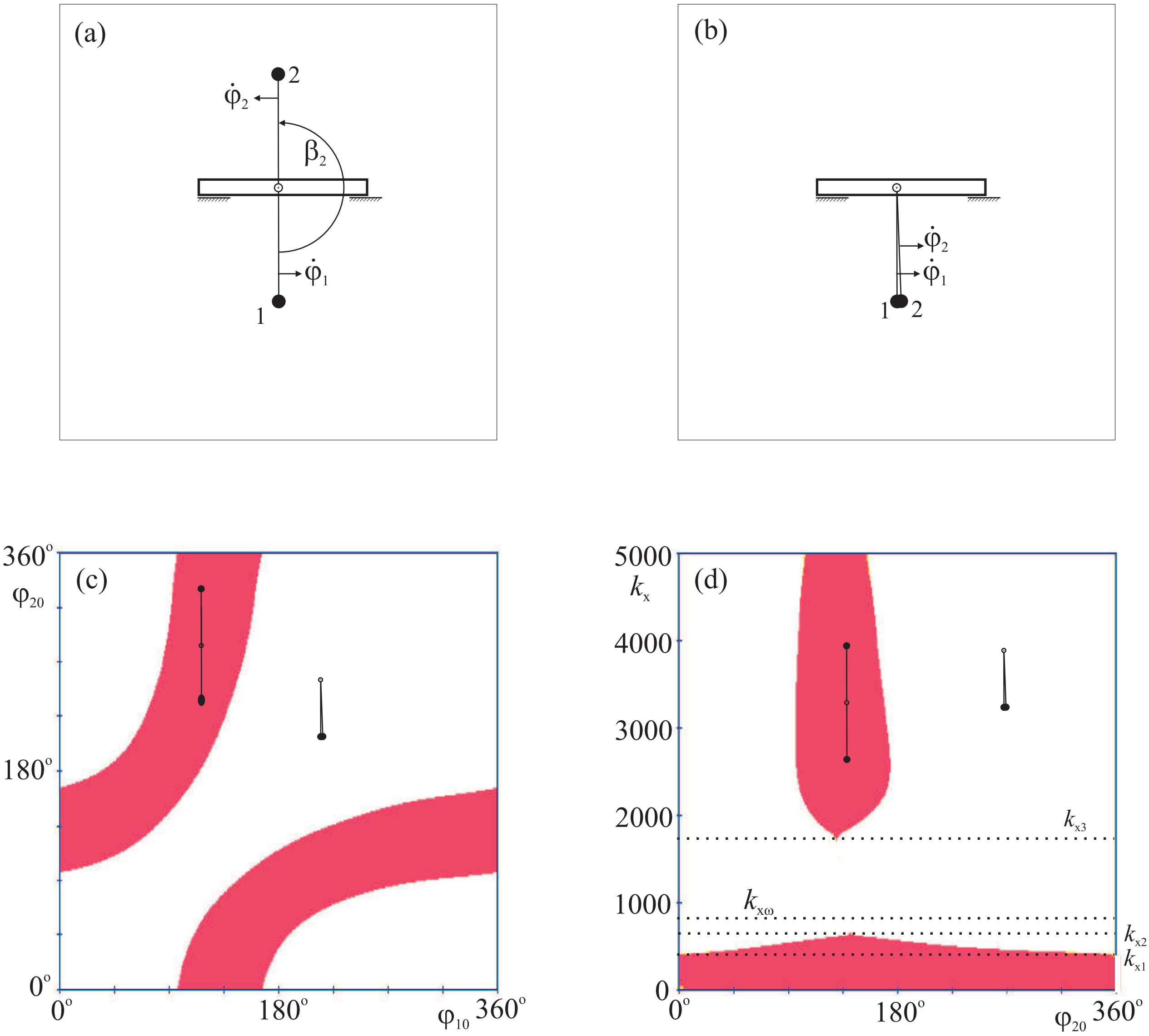

Fig. 3: The pendulums' configurations characteristic for the system (1-2) with $n = 2$ pendulums and its basins of attraction; (a) configuration of antiphase synchronization with $\beta_1 = 0.0^o$ and $\beta_2 = 180.0^o$, (b) complete synchronization, (c) basins of attraction of the complete (white color) and anti-phase (red color) synchronization states, $k_x = 3600.0$ (the basins are shown in the $\varphi_{10} - \varphi_{20}$ plane with fixed initial velocities $\dot{\varphi}_{10} = \dot{\varphi}_{20} = 0.0.$, (d) basins attraction for different values of the stiffness coefficient k_x, shown on the plane $k_x - \varphi_{20}$ ($\varphi_{10} = 0.0^o$, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = 0.0$).

March 2, 2011 11:08 pendulums2

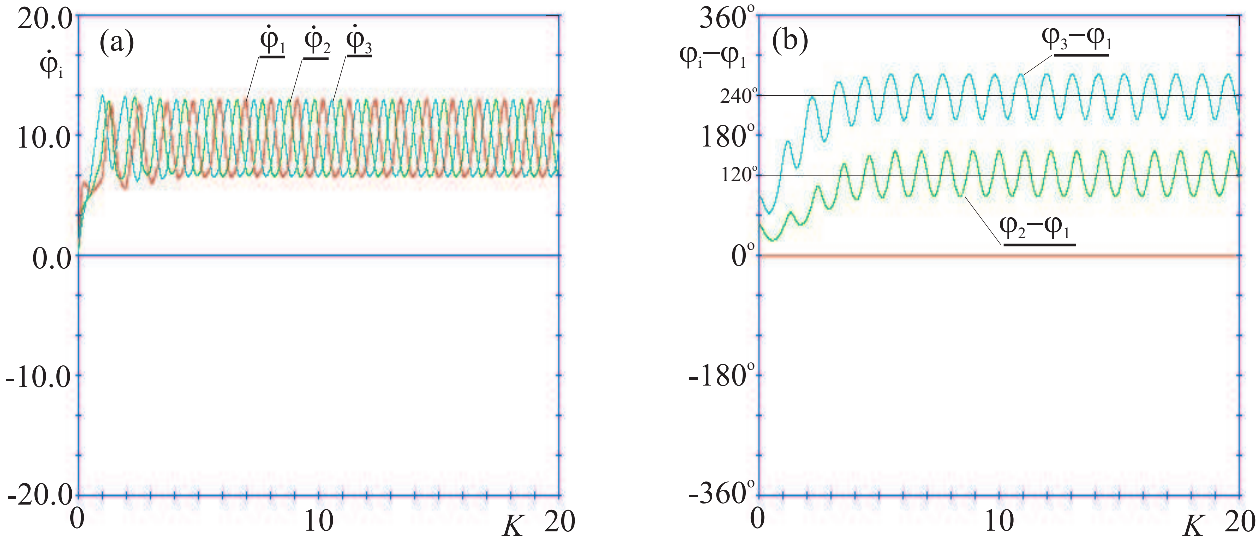

Fig. 4: Time series of pendulums' velocities and displacements in the case of phase synchronization,. (a) angular velocities of pendulums $\dot{\varphi}_1$, $\dot{\varphi}_2$ and $\dot{\varphi}_3$ for a system (1-2) with low stiffness coefficient $k_x = 100.0$ and $\alpha_x = 3.33$, $\varphi_{10} = 0.0^o$, $\varphi_{20} = 45.0^o$, $\varphi_{30} = 90.0^o$, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$ (the unit of time on the horizontal axis is the number $K = \frac{\omega t}{2\pi}$ i.e., the number of complete revolutions of the pendulum rotating with constant angular velocity ω), (b) angular displacement of pendulums $\varphi_2 - \varphi_1$ and $\varphi_3 - \varphi_1$ related to the displacement of the first pendulum, $\beta_2 = 120.0^o$, $\beta_3 = 240.0^o$.

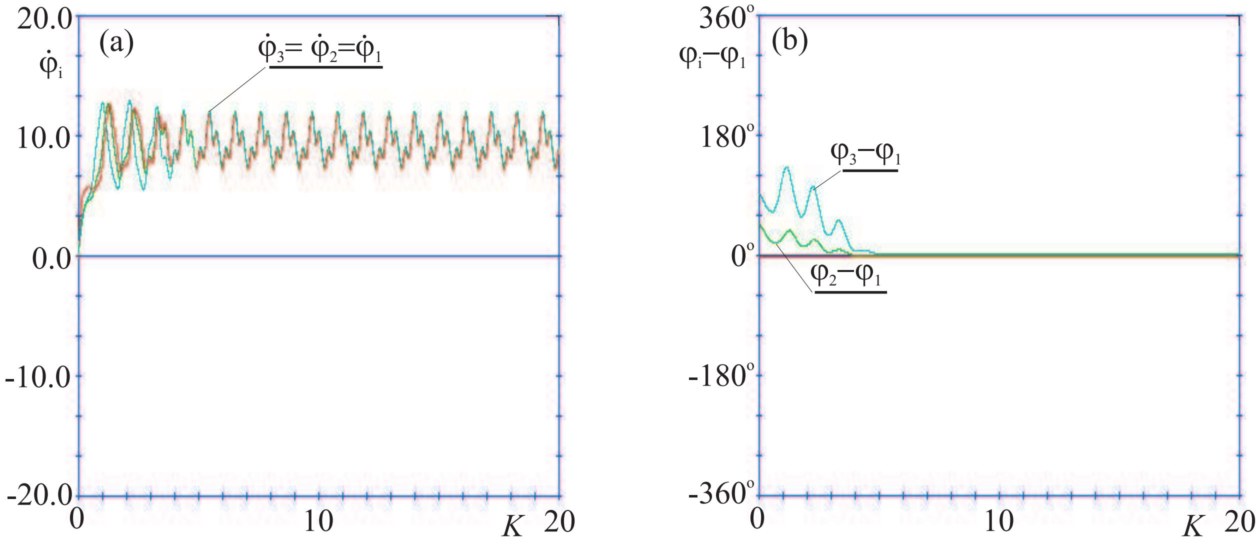

Fig. 5: Time series of pendulums' velocities and displacements in the case of the complete synchronization,. (a) angular velocities of pendulums $\dot{\varphi}_1$, $\dot{\varphi}_2$ and $\dot{\varphi}_3$ for a system (1-2) with low stiffness coefficient $k_x = 3600.0$ and $\alpha_x = 20.0$, $\varphi_{10} = 0.0^o$, $\varphi_{20} = 45.0^o$, $\varphi_{30} = 90.0^o$, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$ (the unit of time on the horizontal axis is the number $K = \frac{\omega t}{2\pi}$ i.e., the number of complete revolutions of the pendulum rotating with constant angular velocity ω), (b) angular displacement of pendulums $\varphi_2 - \varphi_1$ and $\varphi_3 - \varphi_1$ related to the displacement of the first pendulum, $\beta_2 = \beta_3 = 0.0^o$

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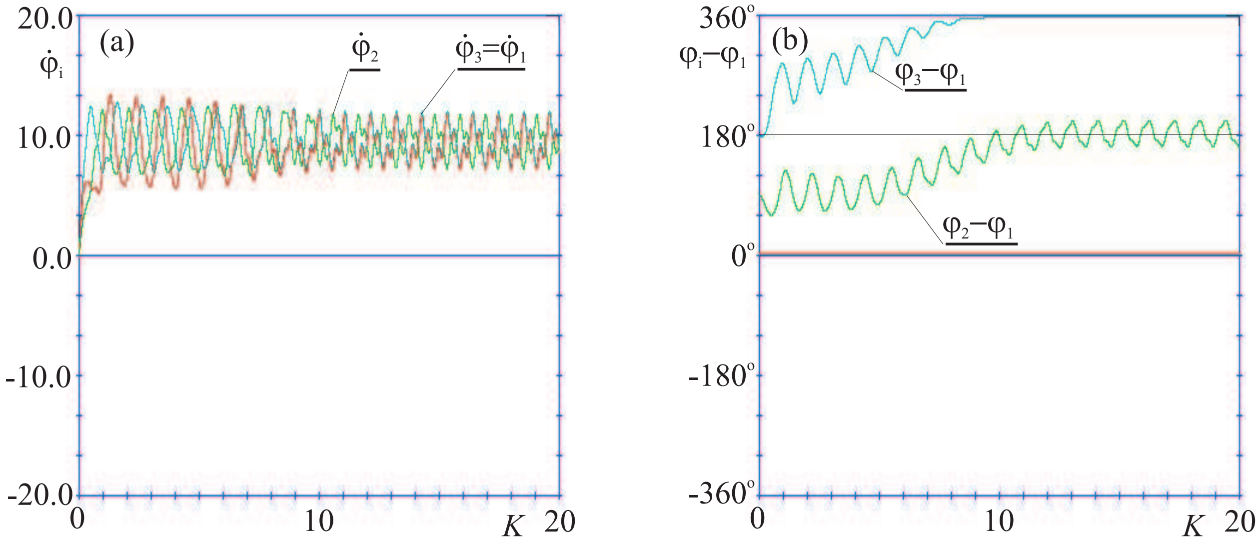

Fig. 6: Time series of pendulums' velocities and displacements in the case of the antiphase synchronization of pendulum 2 with a cluster consisting of pendulums 1 and 3; (a) angular velocities of pendulums $\dot{\varphi}_1$, $\dot{\varphi}_2$ and $\dot{\varphi}_3$ for a system (1-2) with low stiffness coefficient $k_x = 3600.0$ and $\alpha_x = 20.0$, $\varphi_{10} = 0.0^o$, $\varphi_{20} = 90.0^o$, $\varphi_{30} = 180.0^o$, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$ (the unit of time on the horizontal axis is the number $K = \frac{\omega t}{2\pi}$ i.e., the number of complete revolutions of the pendulum rotating with constant angular velocity ω), (b) angular displacement of pendulums $\varphi_2 - \varphi_1$ and $\varphi_3 - \varphi_1$ related to the displacement of the first pendulum, β_2=180.0^o, β_3=360.0^o.

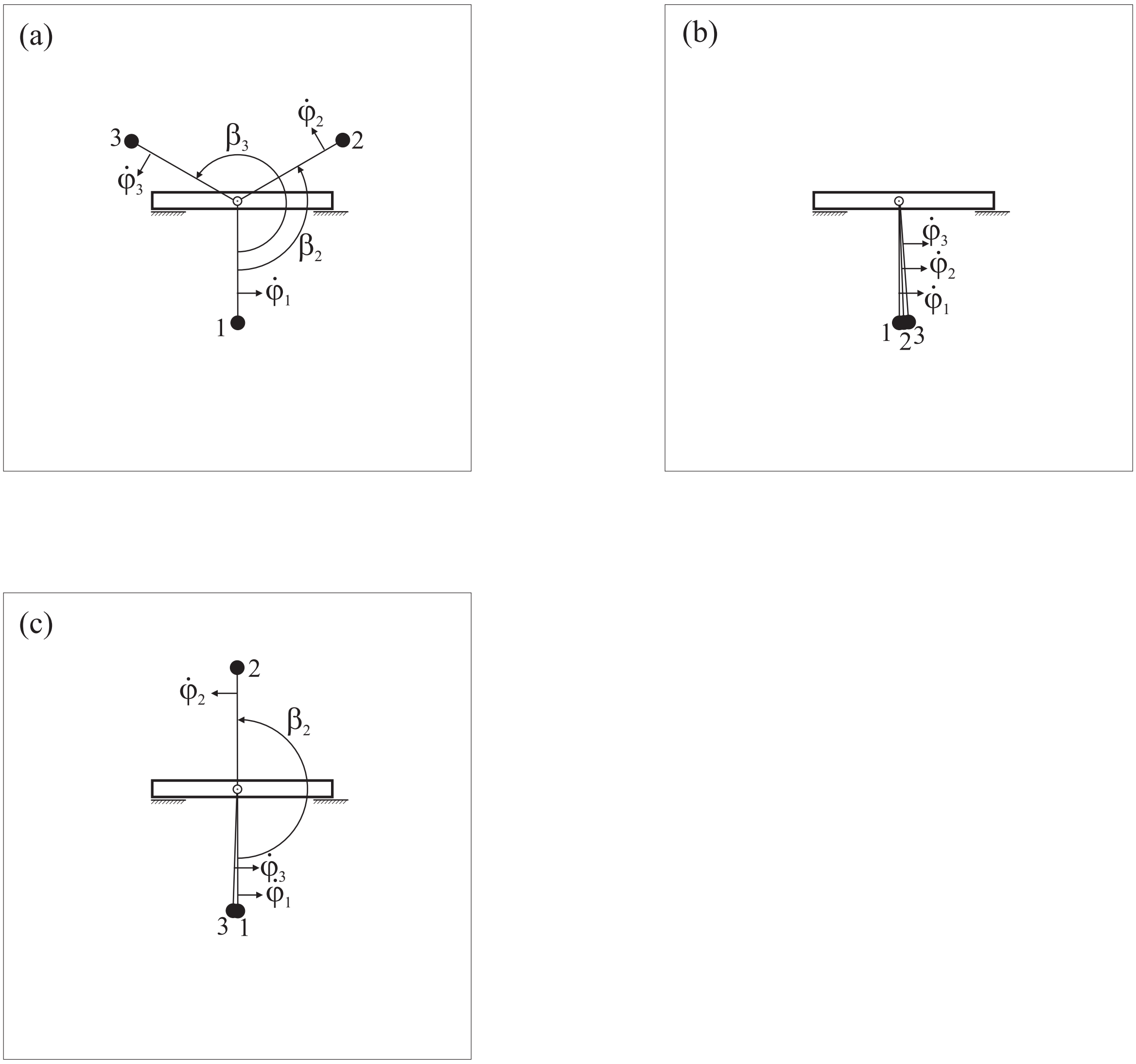

Fig. 7: Synchronization configurations in the system (1-2) with n=3 pendulums; (a) phase synchronization with phase shifts between pendulums: β_1=0.0^o, β_2=120.0^o i β_3=240.0^o, (b) complete synchronization ($\beta_1 = \beta_2 = \beta_3$=0), (c) antiphase synchronization of a single pendulum with the cluster of two other pendulums, β_1=0.0^o, β_2=180.0^o i β_3=0.0^o.

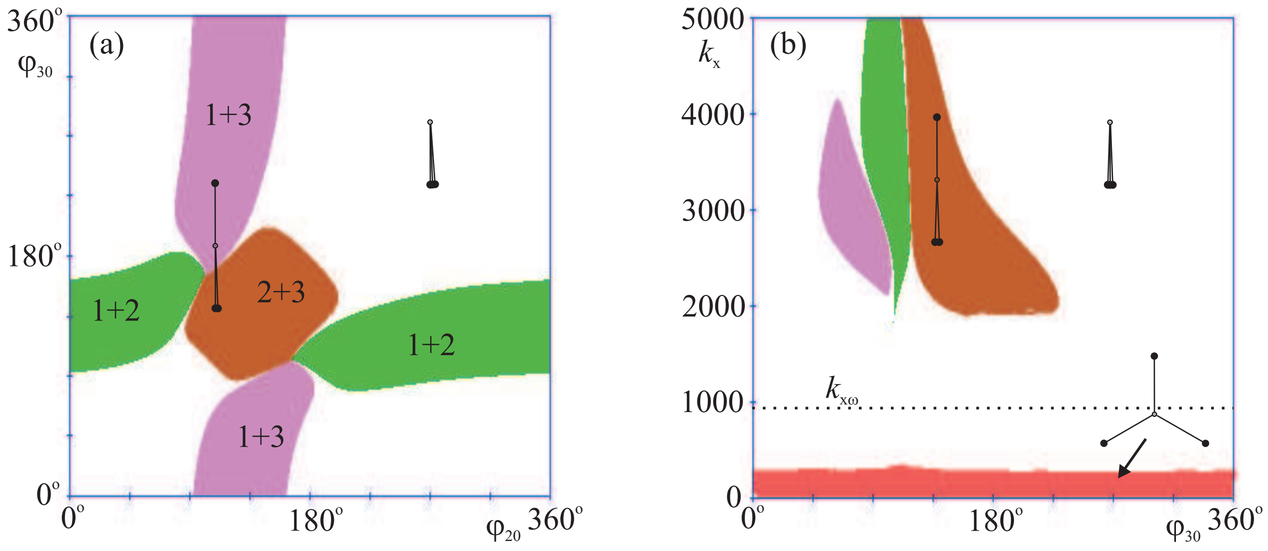

Fig. 8: Basins of attraction of the different states of pendulums' synchronization, (a) basins of attraction of complete (white color) and anti-phase (brown, green, pink colors for different pairs of pendulums in the cluster) synchronization states for a system (1-2), $k_x = 3600.0$ (the basins are shown in the $\varphi_{20} - \varphi_{30}$plane, φ_{10}=0.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$; (b) basins of attraction of complete (white color), anti-phase (brown, green, pink colors for different pairs of pendulums in the cluster) and phase (red color at the bottom) for different values of stiffness coefficient k_x , φ_{10}=0.0^o, φ_{20}=180.0^o, $\dot{\varphi}_{10} = \dot{\varphi}_{20} = \dot{\varphi}_{30} = 0.0$.

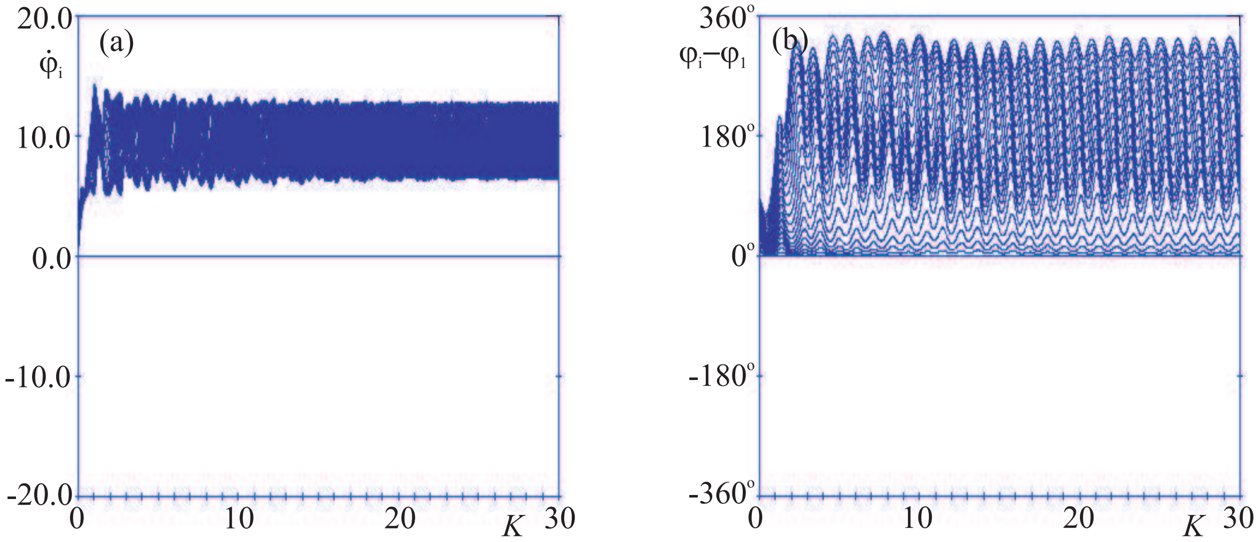

Fig. 9: Phase synchronization of $n = 20$ pendulums in the system (1-2):$m_1 = m_2 = \ldots = m_{20} = 1.00$, $l = 0.25$, $c_\varphi = 0.01$, $N_0 = 5.00$, $N_1 = 0.50$, $k_x = 1000.0$, $M = 20.0$, $\varphi_{i0} = \frac{i \times 90^o}{20}$, $\dot{\varphi}_{i0} = 0.0$; (a) pendulums' velocities $\dot{\varphi}_i$, (b) angular displacements $\varphi_i - \varphi_1$.

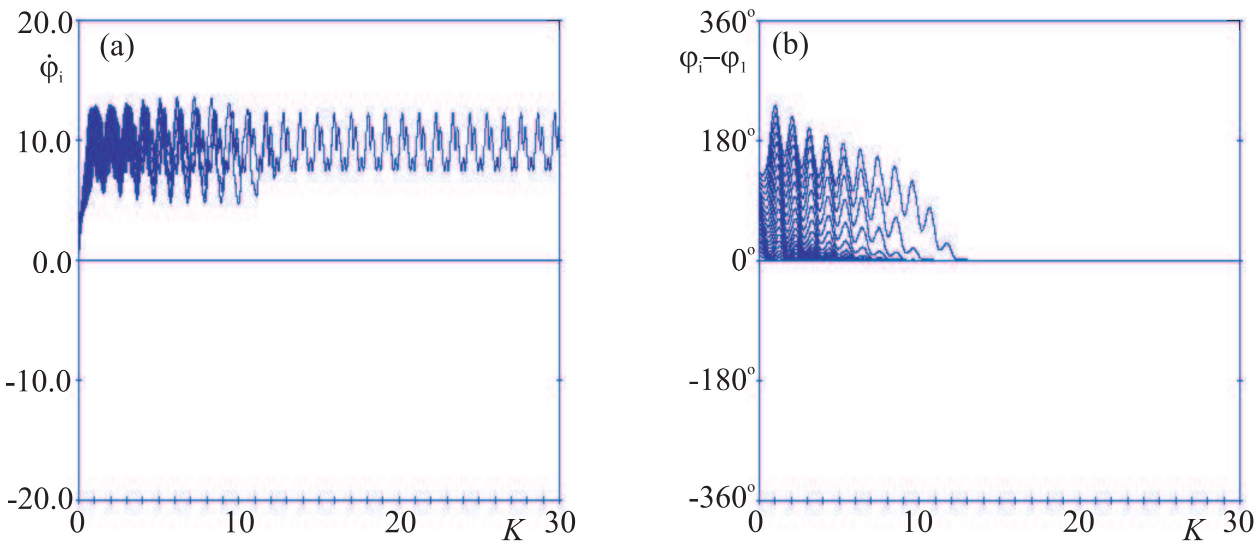

Fig. 10: Complete synchronization of 20 pendulums in system (1-2): $m_1 = m_2 = \ldots = m_{20} = 1.00$, $l=0.25$, $c_\varphi=0.01$, $N_0 = 5.00$, $N_1 = 0.50$, $k_x = 20000.0$, $M = 20.0$, $\varphi_{i0} = \frac{i \times 140^o}{20}$, $\dot{\varphi}_{i0} = 0.0$; (a) pendulums velocities $\dot{\varphi}_i$, (b) angular displacements $\varphi_i - \varphi_1$

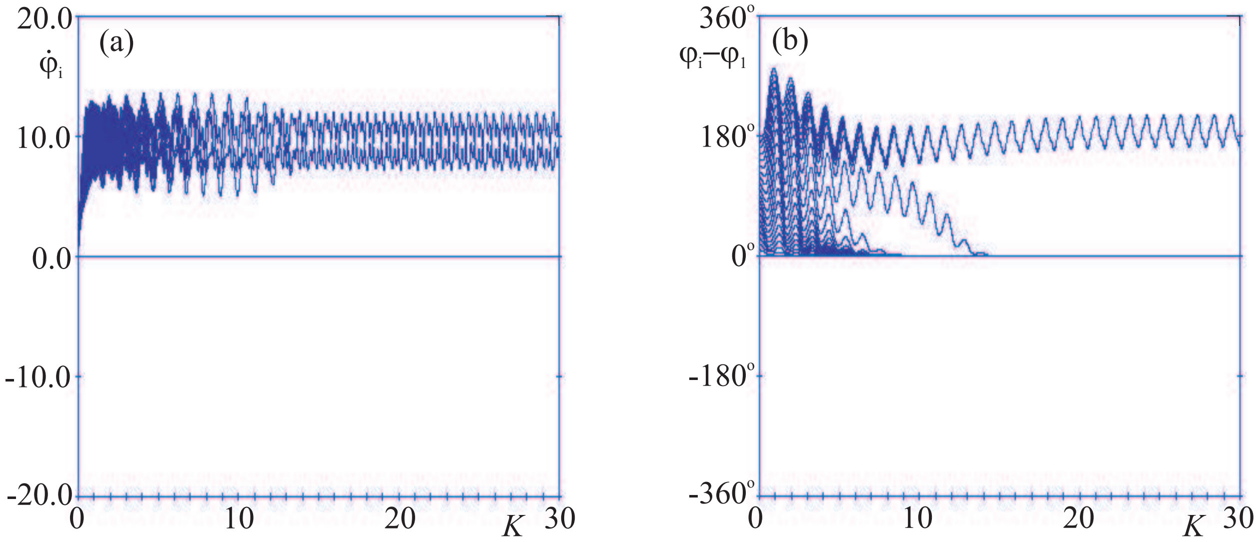

Fig. 11: Cluster synchronization of 20 pendulums in system (1-2): $m_1 = m_2 = \ldots = m_{20} = 1.00$, $l = 0.25$, $c_\varphi = 0.01$, $N_0 = 5.00$, $N_1 = 0.50$, $k_x = 20000.0$, $M = 20.0$ $\varphi_{i0} = \frac{i \times 180^o}{20}$, $\dot{\varphi}_{i0} = 0.0$; (a) pendulums velocities $\dot{\varphi}_i$, (b) angular displacements $\varphi_i - \varphi_1$. Clusters of 7 and 13 pendulums are synchronized in antiphase.