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Clustering of Huygens' Clocks

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We study synchronization of a number of pendulum clocks hanging from an elastically fixed horizontal beam. It has been shown that after a transient, different types of synchronization between pendulums can be observed; (i) the complete synchronization in which all pendulums behave identically, (ii) pendulums create three or five clusters of synchronized pendulums, (iii) anti-phase synchronization in pairs (for even n). We give evidence why the configurations with a different number of clusters are not observed.

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In the 17th century the Dutch researcher Christian Huygens showed that a couple of mechanical clocks hanging from a common support were synchronized.[1)] Over the last three decades the subject of the synchronization has attracted the increasing attention from different fields.[2)–4)] In Huygens experiment[1),5)–9)] clocks (subsystems) are coupled thought elastic structure. Generally, this type of coupling allows investigating how the dynamics of the particular subsystem is influenced by the dynamics of other subsystems.[10)–12)] However, the precise dynamics of the n clocks hanging from the common support is unknown. Here, we study a synchronization problem for n pendulum clocks hanging from an elastically fixed horizontal beam. Each pendulum performs a periodic motion which starts from different initial conditions. We show that after a transient, different types of synchronization between pendulums can be observed; (i) the complete synchronization in which all pendulums behave identically, (ii) pendulums create three or five clusters of synchronized pendulums, (iii) anti-phase synchronization in pairs (for even n). Our results demonstrate that other stable cluster configurations do not exist. We anticipate our assay to be a starting point for further studies of the synchronization and creation of the small-worlds[13)–16)] in the systems coupled by an elastic medium. For example, the behavior of the biological systems (groups of humans or animals) located on elastic structure could be investigated. In particular, a general mechanism for crowd synchrony can be identified.

The large oscillations of London's Millennium Bridge on the day it was opened have restarted the interest in be dynamical behavior of the systems coupled by elastic structure. The detailed theoretical and experimental explanation of the phenomena observed by Huygens for two pendulum clocks has been presented.[5)–9)] In our pre-

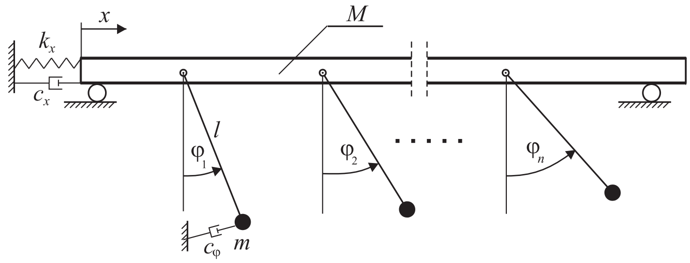

Fig. 1. The model of n pendulums hanging from an elastic horizontal beam.

vious works[10)–12)] we consider the dynamics of Duffing and van der Pol oscillators suspended on the beam and identify a simple mechanism of mutual interaction leading to the periodization of initially chaotic systems. Finally, the assumption of the inseparability of the bridge wobbling and crowd synchrony allows to explain the behavior of Millennium Bridge.[17),18)]

In the current studies we consider a system shown in Fig. 1. The beam of mass M can move in the horizontal direction x with the viscous friction given by a damping coefficient c_x. One side of the beam is attached to the base through the spring with stiffness coefficient k_x. The beam supports n identical pendulum clocks with pendulums of the length l and mass m. The position of the i-th pendulum is given by a variable ϕ_i and its oscillations are damped by viscous friction described by damping coefficient c_ϕ.

In the absence of damping and driving, the Lagrangian of the considered system is

$$L = \frac{1}{2}(M+nm)\dot{x}^2 + m\dot{x}l\sum_{i=1}^{n}\cos\phi_i\dot{\phi}_i + \frac{1}{2}ml^2\sum_{i=1}^{n}\dot{\phi}_i^2 + mgl\sum_{i=1}^{n}\cos\phi_i - \frac{1}{2}kx^2, \quad (1)$$

where $i = 1, 2, \ldots, n$ and g is a gravitational acceleration.

The system equations can be written in a form of Euler-Lagrange equations:

$$\begin{aligned} &ml^2\ddot{\phi}_i + c_\phi\dot{\phi}_i + m\ddot{x}l\cos\phi_i + mgl\sin\phi_i = M_i, \\ &(M+nm)\ddot{x} + \sum_{i=1}^{n}(ml\ddot{\phi}_i\cos\phi_i - ml\dot{\phi}_i^2\sin\phi_i) + c_x\dot{x} + k_x x = 0\,. \end{aligned} \quad (2)$$

The clock escapement mechanism represented by M_i provides the energy needed to compensate the energy dissipation due to the viscous friction and to keep the pendulum running. (The details on the escapement mechanisms can be found in Refs. 19) and 20). This mechanism acts in two successive steps (the first step is followed by the second one and the second one by the first one). In the first step, if $0 < \phi_i < \gamma_N$, then $M_i = M_N$ and when $\phi_i < 0$, then $M_i = 0$, where γ_N and M_N are constant values which characterize the mechanism. For the second stage one has for $-\gamma_N < \phi_i < 0$ $M_i = -M_N$ and for $\phi_i > 0$ $M_i = 0$. Under these assumptions dynamics of the pendulum clock is described by a self-excited oscillator with a limit cycle[21)] (see also Refs. 7)–9)).

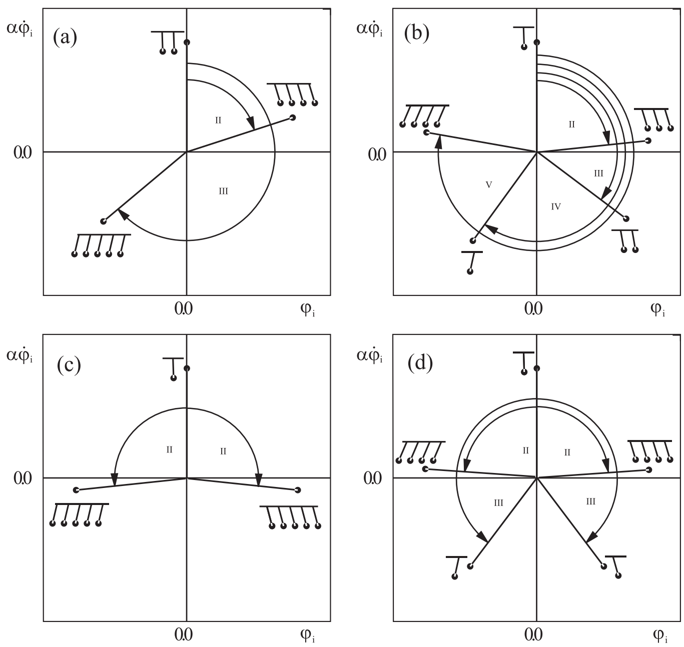

Fig. 2. Cluster configurations for $n = 11$ pendulums; $m = 1$, $l = g/4\pi^2$, $c_\phi = 0.01$, $M = 1$, $c_x = 0.1M$, $k_x = M$, $M_N = 0.075$, $\gamma_N = \pi/36$: (a) three cluster configuration; $n_I = 2$, $n_{II} = 4$, $n_{III} = 5$, (b) five cluster configuration; $n_I = 1$, $n_{II} = 3$, $n_{III} = 2$, $n_{IV} = 1$, $n_V = 4$, (c) symmetrical three cluster configuration; $n_I = 1$, $n_{II} = 5$, $n_{III} = 5$, (d) symmetrical five cluster configuration; $n_I = 1, n_{II} = 4$, $n_{III} = 1$, $n_{IV} = 1$, $n_V = 4$.

In the case of the small oscillations one can assume that after the initial transients the pendulums perform periodic limit cycle oscillations which can be approximately described as

$$\phi_i = \Phi \sin(\alpha t + \beta_i), \tag{3}$$

where Φ, α and β_i are respectively the amplitude, the frequency and the phase difference. As all pendulums are the same, so Φ and α are the same for each of them. The oscillations of the pendulums differ only by the phase difference β_i.[*)]

In our numerical simulations we consider $n \leq 30$, $m = 1$, $l = g/4\pi^2$, and $c_\phi = 0.01$, so the frequency of pendulums oscillations α is equal to 2π. We assume that the initial conditions for pendulums are given by the initial value of β_{i0}, i.e., $\phi_{i0} = \Phi \sin \beta_{i0}$ and $\dot{\phi}_{i0} = \alpha\Phi \cos \beta_{i0}$. The stiffness k_x and damping c_x coefficients are assumed to be proportional to the beam mass M.

In Fig. 2 we plot the position of each of $n = 11$ pendulums in the phase space $\phi_i, \alpha\dot{\phi}_i$ at the time when the first pendulum is moving through the equilibrium position $\phi_1 = 0$ with the positive velocity $\dot{\phi}_1 > 0$. Figure 2(a) shows configuration of three clusters with respectively $n_I = 2, n_{II} = 4$, and $n_{III} = 5$ pendulums. The pendulums in each cluster are synchronized. Figure 2(b) shows the

*) In numerical simulations of Eqs. (1) and (2) we got: $\phi_i = 0.144 \sin(\alpha t + \beta_i) + 0.0033 \sin 3(\alpha t + \beta_i) + 6.75 * 10^{-4} \sin 5(\alpha t + \beta_i) + 3.2 * 10^{-4} \sin 7(\alpha t + \beta_i) + \ldots$ which clearly shows that the higher harmonics are small and have small (negligible) influence on the system (2) motion.

configuration with five clusters of respectively $n_I = 1, n_{II} = 3, n_{III} = 2, n_{IV} = 1$, and $n_V = 4$ pendulums. In Fig. 2(c) the special case of the symmetrical three clusters configuration with respectively $n_I = 1, n_{II} = 5$, and $n_{III} = 5$ pendulums is shown. Figure 2(d) presents the symmetrical configuration of five clusters $n_I = 1, n_{II} = 4, n_{III} = 1, n_{IV} = 1$, and $n_V = 4$ pendulums. For all considered n besides the above configurations we have observed; (i) a complete synchronization, (ii) desynchronous behavior of all pendula. In desynchronous regime the phase difference between pendulums is not constant and is changing chaotically. Additionally, for even n anti-phase synchronization in pairs (for even n) is observed.

To study the stability of the observed steady states we add perturbations δ_i and σ to the variables ϕ_i and x in Eq. (2) and obtain the following linearized variational equation:

$$\begin{aligned} & ml^2\ddot{\delta}_i + m\ddot{\sigma}l\cos\phi_i + ml\delta_i(g\cos\phi_i - \ddot{x}\sin\phi_i) + c_\phi\dot{\delta}_i = 0\ , \\ & (M+nm)\ddot{\sigma} + \sum_{i=1}^{n}(ml\ddot{\delta}_i\cos\phi_i - ml\dot{\phi}_i^2\delta_i\cos\phi_i - ml\ddot{\phi}_i\sin\phi_i - 2ml\dot{\phi}_i\dot{\delta}_i\sin\phi_i) \\ & \qquad + c_x\dot{\sigma} + k_x\sigma = 0\ . \end{aligned} \tag{4}$$

The solution of Eq. (2) given by $\phi_i(t)$ and $x(t)$ is stable when the solution of Eq. (4) δ_i and σ tend to zero for $t \to \infty$. All observed steady states are stable in a wide range of the control parameter M as the solution of the variational equation (4) decays. In the considered range of n we have not observed other stable cluster configuration.

The synchronization between the pendulums can be obtained as a result of the interplay between the period of oscillations of the pendulums, the amplitude of the beam oscillations and the phase difference between the motions of the pendulums and the beam. In the simplest example of one pendulum hanging from the beam, the beam motion which is in phase (out of phase) with motion of the pendulum increases (decreases) the period of oscillations of the pendulum. In the case of n pendulums hanging from the beam, the beam motion can temporarily increase or decrease their period of oscillations allowing synchronization. One should notice here that the described mechanism explains why it is impossible to observe the existence of two groups of pendulums with unequal number of members which synchronize in anti-phase, i.e., due to the unequal total mass of the pendulums in each group the influence of the beam motion on each group cannot be the same. The beam in rest cannot influence this behavior at all. The larger displacement of the beam implies the larger influence of the pendulums behavior. The largest amplitude of the beam oscillation occurs in the case of the complete synchronization of all pendulums. In the case of even n, when we observe $n/2$ pairs of pendulums synchronized in anti-phase the beam is in rest.

To give an explanation why only three and five cluster configurations are observed we consider a horizontal displacement of the beam Under the assumption (3) the second equation of Eq. (2) can be linearized to the following form:

$$(M+nm)\ddot{x} + c_x\dot{x} + k_x x = \sum_{i=1}^{n}(-ml\ddot{\phi}_i + ml\dot{\phi}_i^2\phi_i). \tag{5}$$

Substituting Eq. (3) in Eq. (5) and taking into consideration the relation $\cos^2\alpha\sin\alpha = 1/4(\sin\alpha + 3\sin 3\alpha)$, one gets

$$(M+nm)\ddot{x} + c_x\dot{x} + k_x x = \sum_{i=1}^{n}\Big(ml\alpha^2\Phi\sin(\alpha t+\beta_i) + \frac{1}{4}ml\alpha^2\Phi^3(\sin(\alpha t+\beta_i) + 3\sin 3(\alpha t+\beta_i))\Big). \quad (6)$$

After the initial transient the solution $x(t)$ of Eq. (6) can be expressed in the form

$$x(t) = A_1\sum_{i=1}^{n}\sin(\alpha t+\beta_i+\rho_1) + A_3\sum_{i=1}^{n}\sin 3(\alpha t+\beta_i+\rho_3). \quad (7)$$

where A_1, A_3, ρ_1 and ρ_3 are constant.*)

Besides the local minimum for the $n/2$ pairs of pendulums synchronized in anti-phase (for even n) beam displacement $x(t)$ given by Eq. (7) has local minima in two cases; (i) when the sum of the first harmonic components is equal to zero, (ii) when the sum of the first and the third harmonic components is equal to zero. It is possible to show that the first case occurs for three cluster configuration and the second one for five cluster configuration. The lack of other harmonics components in Eq. (6) shows why the configurations with a number of clusters different from 3 or 5 are not observed.

Three cluster configuration (case (i)) with respectively n_I, n_{II} and n_{III} pendulums in the successive cluster, occurs when angles β_{II} and β_{III} fulfill the relation:

$$\begin{aligned} &n_I + n_{II}\cos\beta_{II} + n_{III}\cos\beta_{III} = 0,\\ &n_{II}\sin\beta_{II} + n_{III}\sin\beta_{III} = 0. \end{aligned} \quad (8)$$

Equation (8) has been obtained from Eq. (7) and the condition for the sum of the first harmonic components to be equal to zero.

For five cluster configuration (case (ii)) with respectively $n_I, n_{II}, n_{III}, n_{IV}$ and n_V pendulums in the successive cluster, exists when angles β_{II-V} fulfill the relation:

$$\begin{aligned} &n_I + n_{II}\cos\beta_{II} + n_{III}\cos\beta_{III} + n_{IV}\cos\beta_{IV} + n_V\cos\beta_V = 0,\\ &n_{II}\sin\beta_{II} + n_{III}\sin\beta_{III} + n_{IV}\sin\beta_{IV} + n_V\sin\beta_V = 0,\\ &n_I + n_{II}\cos 3\beta_{II} + n_{III}\cos 3\beta_{III} + n_{IV}\cos 3\beta_{IV} + n_V\cos 3\beta_V = 0,\\ &n_{II}\sin 3\beta_{II} + n_{III}\sin 3\beta_{III} + n_{IV}\sin 3\beta_{IV} + n_V\sin 3\beta_V = 0. \end{aligned} \quad (9)$$

Equation (9) has been obtained from Eq. (7) and the condition for the sum of the first and third harmonic components to be equal to zero.

In the linear approximation for the three clusters configuration the beam oscillates periodically as forced by the third harmonic component of Eq. (6) as the

*) The exact formulas for the parameters A_1, A_3 and ρ can be found in any textbook on the theory of linear oscillations, e.g., J J. Thomsen, *Vibrations and Stability; Order and Chaos* (McGraw Hill, London, 1997).

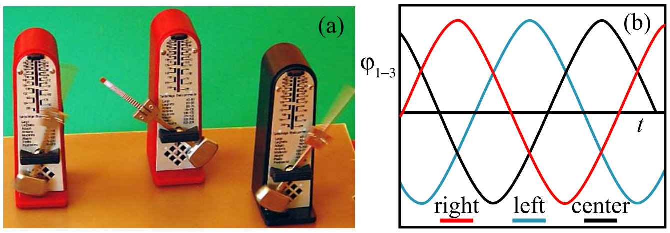

Fig. 3. Three metronomes located on the elastic plate which can roll on the base obtain a synchronous state with a phase difference $\beta_{II} = 2\pi/3, \beta_{III} = 2\pi - \beta_{II}$.

first harmonic components vanishes. For five clusters configuration it stays in rest as both the first and third harmonic components vanish. In numerical simulations of Eq. (2) we observed additional small oscillations due to the higher dimensional harmonics (omitted in the linear approximation) and discontinuous characteristic of forcing given by escapement mechanism).

In Fig. 3 we show the simple experimental confirmation of the stability of symmetrical synchronization of $n = 3$ pendulums. Three metronomes located on the elastic plate which can roll on the base obtain a synchronous state with a phase difference $\beta_{II} = 2\pi/3, \beta_{III} = 2\pi - \beta_{II}$.

To summarize, we have studied the phenomenon of synchronization in the array of the pendulum clocks hanging from an elastically fixed horizontal beam. We show that besides the complete synchronization of all pendulums and creation of the pairs of pendulums synchronized in anti-phase (for even n), the pendulums can be grouped either in three of five clusters. Pendulums in the clusters perform complete synchronization and the clusters are in the form of phase synchronization characterized by a constant phase difference between the pendulums given by Eqs. (8) and (9). We give evidence that the observed behavior is robust in the phase space and can be observed in real experimental systems.

Acknowledgements

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